

Spinorial Wave Equations and Stability of the Milne Spacetime

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Abstract

The spinorial version of the conformal vacuum Einstein field equations are used to construct a system of quasilinear wave equations for the various conformal fields. As a part of analysis we also show how to construct a subsidiary system of wave equations for the zero quantities associated to the various conformal field equations. This subsidiary system is used, in turn, to show that under suitable assumptions on the initial data a solution to the wave equations for the conformal fields implies a solution to the actual conformal Einstein field equations. The use of spinors allows for a more unified deduction of the required wave equations and the analysis of the subsidiary equations than similar approaches based on the metric conformal field equations. As an application of our construction we study the non-linear stability of the Milne Universe. It is shown that sufficiently small perturbations of initial hyperboloidal data for the Milne Universe gives rise to a solution to the Einstein field equations which exist towards the future and has an asymptotic structure similar to that of the Milne Universe.

Keywords: Conformal methods, spinors, wave equations, Milne Universe, global existence

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1 Introduction

The conformal Einstein field equations (CEFE) constitute a powerful tool for the global analysis of spacetimes —see e.g. [6, 5, 7, 9, 10, 11]. The CEFE provide a system of field equations for geometric objects defined on a Lorentzian manifold (\mathcal{M}, g) (the so-called *unphysical spacetime*) which is conformally related to a spacetime $(\tilde{\mathcal{M}}, \tilde{g})$ (the so-called *physical spacetime*) satisfying the (vacuum) Einstein field equations. The metrics g and \tilde{g} are related to each other via a rescaling of the form $g = \Xi^2 \tilde{g}$ where Ξ is the so-called *conformal factor*. The CEFE have the property of being regular at the points where $\Xi = 0$ (the so-called *conformal boundary*) and a solution thereof implies, wherever $\Xi \neq 0$, a solution to the Einstein field equations. The great advantage of the conformal point of view provided by the CEFE is that it allows to recast global problems in the physical spacetime $(\tilde{\mathcal{M}}, \tilde{g})$ as local problems in the unphysical one (\mathcal{M}, g) . The CEFE have been extended to include matter sources consisting of suitable trace-free matter —see e.g. [11, 16, 17]. The CEFE can be expressed in terms of a *Weyl connection* (i.e. a connection which is not metric but nevertheless preserves the conformal structure) to obtain a more general

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system of field equations —the so-called *extended conformal Einstein field equations*, see [12]. In what follows, the conformal field equations expressed in terms of the Levi-Civita connection of the metric g will be known as the *standard CEFE*. The analysis of the present article is restricted to this version of the CEFE. The standard CEFE can be read as differential conditions on the conformal factor and some concomitants thereof: the Schouten tensor, the rescaled Weyl tensor and the components of the unphysical metric g —this version of the equations is known as the *metric CEFE*. Alternatively, by supplementing the field equations with the Cartan structure equations, one can replace the metric components by the coefficients of a frame and the associated connection coefficients as unknowns. This *frame version* of the equations allows a direct translation of the CEFE into a spinorial formalism —the so-called *spinorial CEFE*.

In view of the tensorial nature of the CEFE, in order to make assertions about the existence and properties of their solutions, it is necessary to derive from them a suitable evolution system to which the theory of hyperbolic partial differential equations can be applied. This procedure is known as a *hyperbolic reduction*. Part of the hyperbolic reduction procedure consists of a specification of the gauge inherent to the equations. A systematic way of proceeding to the specification of the gauge is through so-called *gauge source functions*. These functions are associated to derivatives of the field unknowns which are not determined by the field equations. This idea can be used to extract a first order symmetric hyperbolic system of equations for the field unknowns for the metric, frame and spinorial versions of the standard CEFE. More recently, it has been shown that gauge source functions can be used to obtain, out of the metric conformal field equations, a system of quasilinear wave equations —see [18]. This particular construction requires the specification of a *coordinate gauge source function* and a *conformal gauge source function* and is close, in spirit, to the classical treatment of the Cauchy problem in General Relativity in [4] —see also [3].

In the present article we show how to deduce a system of quasilinear wave equations for the unknowns of the spinorial CEFE and analyse its relation to the original set of field equations. The use of the spinorial CEFE (or, in fact, the frame CEFE) gives access to a wider set of gauge source functions consisting of *coordinate, frame and conformal gauge source functions*. Another advantage of the spinorial version of the CEFE is that they have a much simpler algebraic structure than the metric equations. In fact, one of the features of the spinorial formalism simplifying our analysis is the use of the symmetric operator $\square_{AB} \equiv \nabla_{Q(A} \nabla_{B)}^Q$ instead of the usual commutator of covariant derivatives $[\nabla_a, \nabla_b]$. As shown in this article, the use of spinors allows a more unified and systematic discussion of the construction of the wave equations and the so-called *subsidiary system* —needed to show that under suitable conditions a solution to the wave equations implies a solution to the CEFE. As already mentioned, in the spinorial formulation of the CEFE the metric is not part of the unknowns. This observation is important since, whenever the wave operator $\square \equiv \nabla_a \nabla^a$ is applied to any tensor of rank one or more, there will appear derivatives of the connection which, in terms of the metric, represent second order derivatives. Thus, if the metric is part of the unknowns, the principal part of the operator \square is altered by the presence of these derivatives. This is an extra complication that needs to be taken into account in the analysis of [18].

The construction of wave equations to the fields of the CEFE gives access to a set of methods of the theory of partial differential equations alternative to that used for first order symmetric hyperbolic systems —see e.g. [22] for a discussion on this. For example, the analysis in [18] is motivated by the analysis of the characteristic problem on a cone for which a detailed theory is available for quasilinear wave equations.

As an application of the hyperbolic reduction procedure put forward in the present article, we provide an analysis of the non-linear stability of the Milne Universe. The Milne Universe is a spatially flat Friedman-Lemaître-Robertson-Walker (FLRW) solution to the Einstein field equations with vanishing cosmological constant —see e.g. [13]. The Milne Universe can be seen to be a part of the Minkowski spacetime written in comoving coordinates adapted to the world line

of a particle. Accordingly, analysing the non-linear stability of the Milne Universe is essentially equivalent to obtaining a proof of the semiglobal stability of the Minkowski spacetime —see [9]. The stability of the Milne Universe has been analysed by different methods in [2] —see also [1]. In our case the stability result follows from the theory of quasilinear wave equations, in particular the property of Cauchy stability, as given in [15]. In broad terms, our stability result for the Milne Universe can be phrased as:

Theorem. *Initial data for the conformal wave equations close enough to the data for the Milne Universe give rise to a solution to the Einstein field equations which exist globally to the future and has the asymptotic structure similar to that of the Milne Universe.*

Outline of the article

This article is organised as follows: in Section 2 we briefly present the spinorial version of the conformal Einstein field equations which will be the starting point of the analysis. In Section 3, we give the derivation of the wave equations. For the sake of clarity, we first present a general procedure and then discuss the peculiarities of each equation. In Section 4, we derive the subsidiary system and analyse the propagation of the constraints. In Section 5, we give a semiglobal existence and stability result for the Milne spacetime as an application of the equations derived in Section 3. In Section 6 some concluding remarks are presented. In order to ease the presentation of the article, we have moved part of the calculations to a series of appendices. In Appendix A we recall some general spinor relations and deduce the general form for the spinorial Ricci identities for a connection which is metric but not necessarily torsion free. In Appendix B a brief discussion of the relation between the transition spinor and the torsion is provided. In Appendix C detailed computations for the subsidiary system are presented. In Appendix D we recall the frame version of the CEFE. In Appendix E an adapted version of a existence and Cauchy stability result for wave equations is given.

Notations and conventions

The signature convention for (Lorentzian) spacetime metrics will be $(+, -, -, -)$. In what follows $\{a, b, c, \dots\}$ will be used as tensor indices and $\{a, b, c, \dots\}$ will be used as spacetime frame indices taking the values $0, \dots, 3$. In this manner, given a basis $\{e_a\}$ a tensor will be denoted with T_{ab} while its components in the given basis will be denoted by $T_{ab} \equiv T_{ab} e_a^a e_b^b$. Most of the analysis will require the use of spinor and $\{A, B, C, \dots\}$ will denote frame spinorial indices with respect to some specified spin dyad $\{\delta_A^A\}$. We will follow the conventions and notation of Penrose & Rindler [21]. In addition, $D^+(\mathcal{A})$, $H(\mathcal{A})$, $J^+(\mathcal{A})$ and $I^+(\mathcal{A})$ will denote the future domain of dependence, the Cauchy horizon, Causal and Chronological future of \mathcal{A} , respectively —see [14, 25].

2 The conformal Einstein field equations in spinorial form

In [19] Penrose introduced a geometric technique to study the far fields of isolated gravitational systems in which, given a spacetime $(\tilde{\mathcal{M}}, \tilde{g}_{ab})$ satisfying the Einstein field equations (*the physical spacetime*), we consider another spacetime (\mathcal{M}, g_{ab}) (*the unphysical spacetime*) such that

$$g_{ab} = \Xi^2 \tilde{g}_{ab} \tag{1}$$

where Ξ is a scalar field (*the conformal factor*). If we try to derive an equation for the unphysical metric g_{ab} using the Einstein field equations

$$\tilde{R}_{ab} = \lambda \tilde{g}_{ab}$$

making the straightforward computation using the conformal transformation rules for the curvature tensors implied by (1) we will obtain an expression which is singular at the conformal

boundary, i.e. the points for which $\Xi = 0$. An approach to deal with this problem was given in [6] where a regular set of equations for the unphysical metric was derived. These equations are known as the *conformal Einstein field equations* (CEFE). In order to recall the (vacuum) spinorial version of these equations let us introduce a frame $\{e_{AA'}\}$ and an associated spinorial dyad $\{\epsilon_A{}^A\}$. Let $\Gamma_{AA'}{}^{CC'}{}_{DD'}$ denote the spinorial counterpart of the connection coefficients of the Levi-Civita connection ∇ respect to the metric g . This spinor can be decomposed as

$$\Gamma_{AA'}{}^{CC'}{}_{BB'} \equiv \Gamma_{AA'}{}^C{}_B \delta_{B'}{}^{C'} + \bar{\Gamma}_{AA'}{}^{C'}{}_{B'} \delta_B{}^C$$

where

$$\Gamma_{AA'}{}^C{}_B \equiv \frac{1}{2} \Gamma_{AA'}{}^{CQ'}{}_{BQ'}, \quad (2)$$

are the reduced spin connection coefficients. Let $\phi_{ABCD} \equiv \Xi^{-1} \Psi_{ABCD}$ represent the so-called *rescaled Weyl spinor* — Ψ_{ABCD} is the Weyl spinor and Ξ is the conformal factor. In the following $L_{AA'BB'}$ will denote the *spinorial counterpart of the Schouten tensor* of the metric g . Likewise, s will denote a concomitant of the conformal factor defined by $s \equiv \frac{1}{4} \square \Xi - \Lambda \Xi$ with $\Lambda \equiv -24R$, where R is the Ricci scalar of the metric g . In addition, $R^C{}_{DAA'BB'}$ and $\rho^C{}_{DAA'BB'}$ will represent shorthands for the *geometric curvature* and the *algebraic curvature*, given explicitly as

$$\begin{aligned} R^C{}_{DAA'BB'} &\equiv e_{AA'} (\Gamma_{BB'}{}^C{}_D) - e_{BB'} (\Gamma_{AA'}{}^C{}_D) \\ &\quad - \Gamma_{FB'}{}^C{}_D \Gamma_{AA'}{}^F{}_B - \Gamma_{BF'}{}^C{}_D \bar{\Gamma}_{AA'}{}^{F'}{}_{B'} + \Gamma_{FA'}{}^C{}_D \Gamma_{BB'}{}^F{}_A \\ &\quad + \Gamma_{AF'}{}^C{}_D \bar{\Gamma}_{BB'}{}^{F'}{}_{A'} + \Gamma_{AA'}{}^C{}_E \Gamma_{BB'}{}^E{}_D - \Gamma_{BB'}{}^C{}_E \Gamma_{AA'}{}^E{}_D, \end{aligned} \quad (3a)$$

$$\rho_{ABCC'DD'} \equiv \Psi_{ABCD} \epsilon_{C'D'} + L_{BC'DD'} \epsilon_{CA} - L_{BD'CC'} \epsilon_{DA}. \quad (3b)$$

Now, let us define the following spinors which will be collectively called *zero-quantities*:

$$\begin{aligned} \Sigma_{AA'}{}^{QQ'}{}_{BB'} e_{QQ'} &\equiv [e_{BB'}, e_{AA'}] - \Gamma_{BB'}{}^Q{}_A e_{QA'} - \bar{\Gamma}_{BB'}{}^{Q'}{}_{A'} e_{AQ'} \\ &\quad + \Gamma_{AA'}{}^Q{}_B e_{QB'} + \bar{\Gamma}_{AA'}{}^{Q'}{}_{B'} e_{BQ'}, \end{aligned} \quad (4a)$$

$$\Delta_{CDBB'} \equiv \nabla_{(C}{}^{Q'} L_{D)Q'BB'} + \phi_{CDBQ} \nabla^Q{}_{B'} \Xi, \quad (4b)$$

$$\Lambda_{BB'CD} \equiv \nabla^Q{}_{B'} \phi_{ABCQ}, \quad (4c)$$

$$Z_{AA'BB'} \equiv \nabla_{AA'} \nabla_{BB'} \Xi + \Xi L_{AA'BB'} - s \epsilon_{AB} \epsilon_{A'B'}, \quad (4d)$$

$$\Xi^C{}_{DAA'BB'} \equiv R^C{}_{DAA'BB'} - \rho^C{}_{DAA'BB'}, \quad (4e)$$

$$Z_{AA'} \equiv \nabla_{AA'} s + L_{AA'CC'} \nabla^{CC'} \Xi. \quad (4f)$$

In terms of these definitions, the *spinorial version of the CEFE* can be succinctly expressed as

$$\Sigma_{AA'}{}^{QQ'}{}_{BB'} e_{QQ'} = 0, \quad \Delta_{CDBB'} = 0, \quad \Lambda_{BB'CD} = 0, \quad (5a)$$

$$Z_{AA'BB'} = 0, \quad \Xi^C{}_{DAA'BB'} = 0, \quad Z_{AA'} = 0. \quad (5b)$$

Remark. Observe that all the equations are first order except for the fourth equation, which can be written as a first order equation by defining $\Sigma_{AA'} \equiv \nabla_{AA'} \Xi$. Also, notice that the trace $Z = Z_{AA'}{}^{AA'}$ renders the definition of the field s . More details about the CEFE and its derivation can be found in [8, 11].

3 The spinorial wave equations

In this section a set of wave equations is derived from the spinorial version of the CEFE. Since the approach for obtaining the equations is similar for most of the zero-quantities, we first provide a general discussion of the procedure. In the subsequent parts of this section we address the peculiarities of each equation. The results of this section are summarised in Proposition 1.

3.1 General procedure for obtaining the wave equations

Before deriving each of the wave equations let us illustrate the general procedure with a model equation. To this end consider an equation of the form

$$\nabla^E_{A'} N_{EA\kappa} = 0, \quad (6)$$

where $N_{EA\kappa} \equiv \nabla_{(E}{}^{B'} M_{A)B'\kappa}$ and κ is an arbitrary string of spinor indices. We can exploit the symmetries using the following decomposition of a spinor of the same index structure

$$T_{EA\kappa} = T_{(EA)\kappa} + \frac{1}{2} \epsilon_{EA} T_Q{}^Q{}_{\kappa},$$

and recast $N_{EA\kappa}$ as

$$N_{EA\kappa} = \nabla_E{}^{B'} M_{AB'\kappa} + \frac{1}{2} \epsilon_{EA} \nabla^Q{}^{B'} M_{QB'\kappa}.$$

Observe that the model equation (6) determines the symmetrised derivative $\nabla_{(E}{}^{B'} M_{A)B'\kappa}$, while the divergence $\nabla^Q{}^{B'} M_{QB'\kappa}$ can be freely specified. Thus, let $F_\kappa(x) \equiv \nabla^Q{}^{B'} M_{QB'\kappa}$ be a smooth but otherwise arbitrary spinor. This spinor, encoding the freely specifiable part of $N_{EA\kappa}$, is the *gauge source function* for our model equation. Taking this discussion into account, the model equation can be reexpressed as

$$\begin{aligned} \nabla^E_{A'} N_{EA} &= \nabla^E_{A'} \nabla_E{}^{B'} M_{AB'\kappa} + \frac{1}{2} \nabla_{AA'} F_\kappa(x) \\ &= \nabla_{E(A'} \nabla_{B')}^E M_{A}{}^{B'}{}_{\kappa} + \frac{1}{2} \epsilon_{A'B'} \nabla_{EQ'} \nabla^{EQ'} M_A{}^{B'}{}_{\kappa} + \frac{1}{2} \nabla_{AA'} F_\kappa(x) = 0 \end{aligned} \quad (7)$$

where in the second row we have used, again, the decomposition of a 2-valence spinor in its symmetric and trace parts. Finally, recalling the definition of the operators

$$\square \equiv \nabla_{AA'} \nabla^{AA'}, \quad \square_{AB} = \nabla_{Q'(A} \nabla_{B)}{}^{Q'},$$

we rewrite equation (7) as

$$\square M_{AA'\kappa} - 2\square_{A'B'} M_A{}^{B'}{}_{\kappa} - \nabla_{AA'} F_\kappa(x) = 0. \quad (8)$$

The spinorial Ricci identities allow us to rewrite $\square_{A'}{}^{B'} M_{AB'\kappa}$ in terms of the curvature spinors—namely, the Weyl spinor $\Psi_{ABCD} = \Xi\phi_{ABCD}$, the Ricci spinor $\Phi_{ABA'B'}$, and the Ricci scalar $\Lambda = -24R$ —and $M_{AB'\kappa}$.

In the rest of the section we will derive the particular wave equations implied by each of the zero-quantities following an analogous procedure as the one used for the model equation.

3.2 Wave equation for the frame (no-torsion condition)

The zero-quantity $\Sigma_{AA'}{}^{QQ'}{}_{BB'}$ encodes the no-torsion condition. The equation (4a) can be conveniently rewritten introducing an arbitrary frame $\{c_a\}$, which allow us to write $e_{AA'} = e_{AA'}{}^a c_a$. Taking this into account we rewrite the zero-quantity for the no-torsion condition as

$$\Sigma_{AA'}{}^{QQ'}{}_{BB'} e_{QQ'}{}^c = \nabla_{BB'}(e_{AA'}{}^c) - \nabla_{AA'}(e_{BB'}{}^c) - C_a{}^c{}_b e_{AA'}{}^a e_{BB'}{}^b, \quad (9)$$

where $C_a{}^c{}_b$ are the commutation coefficients of the frame, defined by the relation $[c_a, c_b] = C_a{}^c{}_b c_c$. For conciseness, we have introduced the notation $\nabla_{AA'} e_{BB'}{}^c$ which is to be interpreted as a shorthand for the longer expression

$$\nabla_{AA'} e_{BB'}{}^c \equiv e_{AA'}{}^c - \Gamma_{AA'}{}^Q{}_{B'} e_{QB'}{}^c - \bar{\Gamma}_{AA'}{}^{Q'}{}_{B'} e_{BQ'}{}^c.$$

Using the irreducible decomposition of a spinor representing an antisymmetric tensor we obtain that

$$\Sigma_{AA'}{}^{QQ'}{}_{BB'}e_{QQ'}{}^c = \epsilon_{AB}\bar{\Sigma}_{A'B'}{}^c + \epsilon_{A'B'}\Sigma_{AB}{}^c \quad (10)$$

where

$$\Sigma_{AB}{}^c \equiv \frac{1}{2}\Sigma_{(A|D'|}{}^{QQ'}{}_{B)}{}^{D'}e_{QQ'}{}^c$$

is a reduced zero-quantity which can be written in terms of the frame coefficients using equation (9) as

$$\Sigma_{AB}{}^c = \nabla_{(A}{}^{D'}e_{B)D'}{}^c + \frac{1}{2}e_{(A}{}^{D'a}e_{B)D'}{}^b C_a{}^c{}_b.$$

Using the decomposition of a valence-2 spinor in the first term of the right-hand side we get

$$\Sigma_{AB}{}^c = \nabla_A{}^{D'}e_{BD'}{}^c + \frac{1}{2}\epsilon_{AB}\nabla^{PD'}e_{PD'}{}^c + \frac{1}{2}e_{(A}{}^{D'a}e_{B)D'}{}^b C_a{}^c{}_b.$$

Introducing the *coordinate gauge source function* $F^c(x) = \nabla^{PD'}e_{PD'}{}^c$, a wave equation can then be deduced from the condition

$$\nabla^A{}_{E'}\Sigma_{AB}{}^c = 0.$$

Observe that this equation is satisfied if $\Sigma_{AB}{}^c = 0$ —that is, if the corresponding CEFE is satisfied. Adapting the general procedure described in Section 3.1 as required, we get

$$\square e_{BE'}{}^c - 2\square_{E'D'}e_B{}^{D'}{}^c - \nabla_{BE'}F^c(x) - \nabla^A{}_{E'}(e_{(A}{}^{D'a}e_{B)D'}{}^b C_a{}^c{}_b) = 0.$$

Finally, using the spinorial Ricci identities and rearranging the last term we get the following wave equation

$$\square e_{BE'}{}^c - 2e^{QD'}{}^c\bar{\Phi}_{QBE'D'} + 6\Lambda e_{BE'}{}^c - e_{(A}{}^{D'a}e_{B)D'}{}^b\nabla^A{}_{E'}C_a{}^c{}_b - 2C_a{}^c{}_be_{(A}{}^{D'a}\nabla^A{}_{|E'|}e_{B)D'}{}^b - \nabla_{BE'}F^c(x) = 0. \quad (11)$$

3.3 Wave equation for the connection coefficients

The spinorial counterpart of the Riemann tensor can be decomposed as

$$R_{AA'BB'CC'DD'} = R_{ABCC'DD'}\epsilon_{B'A'} + \bar{R}_{A'B'CC'DD'}\epsilon_{BA}.$$

where the *reduced curvature spinor* $R_{ABCC'DD'}$ is expressed in terms of the spin connection coefficients as

$$R_{ABCC'DD'} + \Sigma_{CC'}{}^{QQ'}{}_{DD'}\Gamma_{QQ'AB} = \nabla_{CC'}\Gamma_{DD'AB} - \nabla_{DD'}\Gamma_{CC'AB} + \Gamma_{CC'}{}^Q{}_B\Gamma_{DD'QA} - \Gamma_{DD'}{}^Q{}_B\Gamma_{CC'QA}. \quad (12)$$

In the last equation, $\nabla_{DD'}\Gamma_{CC'AB}$ has been introduced for convenience as a shorthand for the longer expression

$$\nabla_{DD'}\Gamma_{CC'AB} \equiv e_{DD'}\Gamma_{CC'AB} - \Gamma_{DD'}{}^Q{}_C\Gamma_{QC'AB} - \bar{\Gamma}_{DD'}{}^Q{}_C\Gamma_{QC'AB} - \Gamma_{DD'}{}^Q{}_B\Gamma_{CC'AQ}.$$

Now, observe that the zero quantity $\Xi_{ABCC'DD'}$ defined in equation (4e) has the symmetry $\Xi_{ABCC'DD'} = \Xi_{(AB)CC'DD'} = -\Xi_{(AB)DD'CC'}$. Exploiting this fact, the reduced spinors associated to the geometric and algebraic curvatures $R_{ABCC'DD'}$ and $\rho_{ABCC'DD'}$ can be split, respectively, as

$$R_{ABCC'DD'} = \epsilon_{C'D'}R_{ABCD} + \epsilon_{CD}R_{ABC'D'}, \quad \rho_{ABCC'DD'} = \epsilon_{C'D'}\rho_{ABCD} + \epsilon_{CD}\rho_{ABC'D'},$$

where

$$R_{ABCD} = \frac{1}{2}R_{AB(C|E'|D)}{}^{E'}, \quad R_{ABC'D'} = \frac{1}{2}R_{ABE(C'E'D')},$$

are the *reduced geometric curvature spinors*. Analogous definitions are introduced for the algebraic curvature¹. The adjective *geometric* is used here to emphasise the fact that R_{ABCD} and $R_{ABC'D'}$ are expressed in terms of the reduced connection coefficients while ρ_{ABCD} and $\rho_{ABC'D'}$, the reduced algebraic curvature spinors, are written in terms of the Weyl spinor Ψ_{ABCD} and the spinorial counterpart of the Schouten tensor $L_{AA'BB'}$. Together, these two reduced geometric and algebraic curvature spinors give the reduced zero quantities

$$\Xi_{ABCD} = R_{ABCD} - \rho_{ABCD}, \quad \Xi_{ABC'D'} = R_{ABC'D'} - \rho_{ABC'D'}.$$

Remark. Observe that although R_{ABCD} and $R_{ABC'D'}$ are independent, their derivatives are related through the *second Bianchi identity*, which implies that

$$\nabla^C_{D'} R_{ABCD} = \nabla^{C'}_D R_{ABC'D'}.$$

This observation is also true for the algebraic curvature as a consequence of the conformal field equations $\Delta_{CDBB'} = 0$ and $\Lambda_{BB'CD} = 0$ since they encode the second Bianchi identity written as differential conditions on the spinorial counterpart of the Schouten tensor and the Weyl spinor. To verify the last statement, recall that the equation for the Schouten tensor encoded in $\Delta_{CDBB'} = 0$ comes from the frame equation

$$\nabla_a C^a_{bcd} = \nabla_c L_{db} - \nabla_d L_{cb}, \quad (13)$$

which can be regarded as the second Bianchi identity written in terms of the Schouten and Weyl tensors. This can be easily checked, since the last equation is obtained from the substitution of the expression for the Riemann tensor in terms of the Weyl and Schouten tensors (i.e. the algebraic curvature) in the second Bianchi identity. This means that, as long as the conformal field equations $\Delta_{CDBB'} = 0$ and $\Lambda_{BB'CD} = 0$ are satisfied we can write

$$\nabla^C_{D'} \rho_{ABCD} = \nabla^{C'}_D \rho_{ABC'D'}.$$

Therefore, the reduced quantities Ξ_{ABCD} and $\Xi_{ACC'D'}$ are related via

$$\nabla^C_{D'} \Xi_{ABCD} = \nabla^{C'}_D \Xi_{ABC'D'}.$$

Now, we compute explicitly the reduced geometric and algebraic curvature. Recalling the definition of $\rho_{ABCC'DD'}$ in terms of the Weyl spinor and the spinorial counterpart of the Schouten tensor as given in equation (3b) it follows that

$$\rho_{ABCD} = \Psi_{ABCD} + L_{BE'(D} \epsilon^{E'}_{C)A}$$

or, equivalently

$$\rho_{ABCD} = \Xi\phi_{ABCD} + 2\Lambda(\epsilon_{DB}\epsilon_{CA} + \epsilon_{CB}\epsilon_{DA}).$$

Similarly,

$$\rho_{ABC'D'} = \Phi_{ABC'D'}. \quad (14)$$

Computing the reduced version of the geometric curvature from expression (12) we get

$$R_{ABCD} = -\frac{1}{2}\Sigma_{(C|E'|}{}^{QQ'}_{D)}{}^{E'}\Gamma_{QQ'AB} + \nabla_{(C|E'|}\Gamma_{D)}{}^{E'}{}_{AB} + \Gamma_{(C|E'}{}^Q{}_{B|}\Gamma_{D)}{}^{E'}{}_{QA}, \quad (15a)$$

$$R_{ABC'D'} = -\frac{1}{2}\Sigma_{E(C'}{}^{QQ'}_{D')}{}^E\Gamma_{QQ'AB} + \nabla_{E(C'}\Gamma_{D')}{}^E{}_{AB} + \Gamma_{E(C'}{}^Q{}_{B|}\Gamma_{D')}{}^E{}_{QA}. \quad (15b)$$

If the no-torsion condition (9) is satisfied, then the first term in each of the last expressions vanishes. In this manner one obtains an expression for the reduced geometric curvature purely in

¹Observe that in contrast with the split (10) used for the no-torsion condition, the reduced spinors R_{ABCD} and $R_{ABC'D'}$ are not complex conjugate of each other.

terms of the reduced connection coefficients and, in turn, a wave equation from either $\nabla^C_{D'}\Xi_{ABCD}$ or $\nabla^{C'}_D\Xi_{ABC'D'}$. In what follows, for concreteness we will consider

$$\nabla^{C'}_D\Xi_{ABC'D'} = 0.$$

Adapting the procedure described in Section 3.1 and taking into account equations (14) and (15a) one obtains

$$\square\Gamma_{DD'AB} - 2\square_{DE}\Gamma^E_{D'AB} - \nabla_{D'D}F_{AB}(x) + 2\nabla^{C'}_D\Gamma_{E(C'Q|B|}\Gamma^E_{D')QA} = 2\nabla^{C'}_D\Phi_{ABC'D'}. \quad (16)$$

The gauge source function that appears in the last expression is the *frame gauge source function* defined by

$$F_{AB}(x) = \nabla^{PQ'}\Gamma_{PQ'AB}.$$

Using the spinorial Ricci identities to replace $\square_{DE}\Gamma^E_{D'AB}$ in equation(16) and exploiting the symmetry $\Gamma^E_{D'AB} = \Gamma^E_{D'(AB)}$ we get

$$\begin{aligned} \square_{DE}\Gamma^E_{D'AB} &= -3\Lambda\Gamma_{DD'AB} + \Gamma^{EH'}_{AB}\Phi_{D'H'DE} \\ &\quad + 2\Xi\phi_{DEH}(A\Gamma^E_{|D'|}{}^H{}_B) - 2\Gamma_{(A|D'D|B)} - 2\Gamma^E_{D'E(B\epsilon|D|A)}. \end{aligned} \quad (17)$$

Substituting the last expression into (16) we get the wave equation

$$\begin{aligned} \square\Gamma_{DD'AB} - 2(\Gamma^{EH'}_{AB}\Phi_{D'H'DE} - 3\Lambda\Gamma_{DD'AB} + 2\Xi\phi_{DEH}(A\Gamma^E_{|D'|}{}^H{}_B) \\ - 2\Gamma_{(A|D'D|B)} - 2\Gamma^E_{D'E(B\epsilon|D|A)}) + 2\nabla^{C'}_D\Gamma_{E(C'Q|B|}\Gamma^E_{D')QA} \\ - 2\nabla^{C'}_D\Phi_{BAC'D'} - \nabla_{D'D}F_{AB}(x) = 0. \end{aligned}$$

3.4 Wave equation for the Ricci spinor

The zero-quantity defined by equation (4b) is expressed in terms of the spinorial counterpart of the Schouten tensor. The spinor $L_{AA'BB'}$ can be decomposed in terms of the Ricci spinor $\Phi_{AA'BB'}$ and Λ as

$$L_{AA'BB'} = \Phi_{AA'BB'} - \Lambda\epsilon_{AB}\epsilon_{A'B'} \quad (18)$$

—see Appendix A for more details. In the context of the CEFÉ the field Λ can be regarded as a gauge source function. Thus, in what follows we regard the equation $\Delta_{CABB'} = 0$ as differential conditions on $\Phi_{AA'BB'}$.

In order to derive a wave equation for the Ricci spinor we consider

$$\nabla^C_{E'}\Delta_{CDBB'} = 0.$$

Proceeding, again, as described in Section 3.1 and using that $\nabla^C_{E'}\phi_{CDBQ} = 0$ —that is, assuming that the equation encoded in the the zero-quantity Λ_{CDBQ} is satisfied— we get

$$\square L_{DBE'B'} - 2\square_{E'Q'}L_{DB}{}^{Q'}{}_{B'} - \nabla_{DE'}\nabla^{EQ'}L_{EQ'BB'} - 2\phi_{CDBQ}\nabla^C_{E'}\nabla^Q_{B'}\Xi = 0.$$

Using the decomposition (18) and symmetrising in CD we further obtain that

$$\square\Phi_{DBE'B'} - 2\square_{E'Q'}\Phi_{DB}{}^{Q'}{}_{B'} - \nabla_{(D|E'}\nabla^{EQ'}L_{EQ'|B)B'} - 2\phi_{CDBQ}\nabla^C_{E'}\nabla^Q_{B'}\Xi = 0. \quad (19)$$

To find a satisfactory wave equation for the Ricci tensor we need to rewrite the last three terms of equation (19). To compute the third term observe that the second contracted Bianchi identity as in equation (54) and the decomposition of the Schouten spinor given by equation (18) render

$$\nabla^{EQ'}L_{EQ'BB'} = \nabla^{EQ'}\Phi_{EQ'BB'} - \epsilon_{EB}\epsilon_{Q'B'}\nabla^{EQ'}\Lambda = -4\nabla_{BB'}\Lambda.$$

Thus, one finds that

$$\nabla_{(D|E'}\nabla^{EQ'}L_{EQ'|B)B'} = -4\nabla_{E'}(D\nabla_{B)B'}\Lambda. \quad (20)$$

This last expression is satisfactory since, as already mentioned, the Ricci scalar R (or equivalently Λ) can be regarded as a gauge source function —the so-called *conformal gauge source function* [8]. In order to replace the last term of equation (19) we use field equation encoded in $Z_{AA'BB'} = 0$ and the decomposition (18), to obtain

$$\phi_{CDBQ}\nabla^C_{E'}\nabla^Q_{B'}\Xi = -\Xi\phi_{CDBQ}L^C_{E'}{}^Q_{B'} = -\Xi\phi_{CDBQ}\Phi^C_{E'}{}^Q_{B'}. \quad (21)$$

Finally, computing $\square_{E'Q'}\Phi_{DB}{}^{Q'}$ and substituting equations (20) and (21) we conclude that

$$\begin{aligned} \square\Phi_{DBE'B'} - 4\Phi^P_{(B}{}^{Q'}_{|B'|}\Phi_{D)P'E'Q'} + 6\Lambda\Phi_{DBE'B'} - 2\Xi\bar{\phi}_{E'Q'B'H'}\Phi_{DB}{}^{Q'H'} \\ + 4\Lambda\Phi_{DB}{}^{Q'}_{(E'\epsilon_{Q')B'}} + 2\phi_{CDBQ}\Phi^C_{E'}{}^Q_{B'} + 4\nabla_{E'}(D\nabla_{B)B'}\Lambda = 0. \end{aligned} \quad (22)$$

3.5 Wave equation for the rescaled Weyl spinor

Proceeding as in the previous subsections, consider the equation

$$\nabla_D{}^{B'}\Lambda_{BB'AC} = 0. \quad (23)$$

Observe that in this case we do not need a gauge source function since we already have a unsymmetrised derivative in the definition of $\Lambda_{BB'AC}$. Following the procedure described in Section 3.1 we get

$$\square\phi_{ABCD} - 2\square_{DQ}\phi_{ABC}{}^Q = 0.$$

Thus, to complete the discussion we need to calculate $\square_{DQ}\phi_{ABC}{}^Q$. Using the spinorial Ricci identities we obtain

$$\square_{DQ}\phi_{ABC}{}^Q = \Xi\phi_{FQAD}\phi_{BC}{}^{FQ} + \Xi\phi_{FQDB}\phi_{AC}{}^{FQ} + \Xi\phi_{FQCD}\phi_{AB}{}^{FQ} - 6\Lambda\phi_{ABCD}$$

The symmetries of ϕ_{ABCD} simplify the equation since

$$\square_{(D|Q}\phi_{A|B)C}{}^Q = 3\Xi\phi^{FQ}_{(AB}\phi_{CD)FQ} - 6\Lambda\phi_{ABCD}.$$

Taking into account the last expression we obtain the following wave equation for the rescaled Weyl spinor

$$\square\phi_{ABCD} - 6\Xi\phi^{FQ}_{(AB}\phi_{CD)FQ} + 12\Lambda\phi_{ABCD} = 0. \quad (24)$$

Observe that the wave equation for the rescaled Weyl spinor is remarkably simple.

3.6 Wave equation for the field s

Since s is a scalar field, the general procedure described in Section 3.1 does not provide any computational advantage. The required wave equation is derived from considering

$$\nabla^{AA'}Z_{AA'} = 0$$

Explicitly, the last equation can be written as

$$\square s + \nabla^{AA'}\Phi_{ACA'C'}\nabla^{CC'}\Xi + \Phi_{ACA'C'}\nabla^{AA'}\nabla^{CC'}\Xi = 0.$$

Using the the contracted second Bianchi identity (54) to replace the second term and the field equation Z_{ABCD} along with the decomposition (18) to replace the third term we get

$$\square s - \Xi\Phi_{ACA'C'}\Phi^{ACA'C'} - 3\nabla_{CC'}\Lambda\nabla^{CC'}\Xi = 0.$$

3.7 Wave equation for the conformal factor

A wave equation for the conformal factor follows directly from the contraction $Z_{AA'}{}^{AA'}$ and the decomposition (53):

$$\square \Xi = 4(s + \Lambda \Xi).$$

3.8 Summary of the analysis

We summarise the results of this section in the following proposition:

Proposition 1. *Let*

$$F^a(x), \quad F_{AB}(x), \quad \Lambda(x)$$

denote smooth functions on \mathcal{M} such that

$$\nabla^{QQ'} e_{QQ'}{}^a = F^a(x), \quad \nabla^{QQ'} \Gamma_{QQ'}{}^{AB} = F_{AB}(x) \quad \nabla^{QQ'} \Phi_{PQP'}{}^{Q'} = -3\nabla_{PP'} \Lambda(x).$$

If the CEFÉ (5) are satisfied on $\mathcal{U} \subset \mathcal{M}$, then one has that

$$\square e_{BE'}{}^c - 2e^{QD'}{}^c \bar{\Phi}_{QBE'D'} + 6\Lambda e_{BE'}{}^c - e_{(A}{}^{D'a} e_{B)D'}{}^b \nabla^A{}_{E'} C_a{}^c{}_b - 2C_a{}^c{}_b e_{(A}{}^{D'a} \nabla^A{}_{|E'|} e_{B)D'}{}^b - \nabla_{BE'} F^c(x) = 0, \quad (25a)$$

$$\square \Gamma_{DD'}{}^{AB} - 2(\Gamma^{EH'}{}_{AB} \Phi_{D'H'DE} - 3\Lambda \Gamma_{DD'}{}^{AB} + 2\Xi \phi_{DEH(A} \Gamma^E{}_{|D'|}{}^{H|B)} - 2\Gamma_{(A|D'D|B)} - 2\Gamma^E{}_{D'E(B\epsilon|D|A)} + 2\nabla^{C'}{}_{D'} \Gamma_{E(C'}{}^Q{}_{|B|} \Gamma^E{}_{D')}{}^{QA} - 2\nabla^{C'}{}_{D'} \Phi_{BAC'D'} - \nabla_{D'D} F_{AB}(x) = 0, \quad (25b)$$

$$\square \Phi_{DBE'B'} - 4\Phi^P{}_{(B}{}^{Q'}{}_{|B'|} \Phi_{D)P'E'Q'} + 6\Lambda \Phi_{DBE'B'} - 2\Xi \bar{\phi}_{E'Q'B'H'} \Phi_{DB}{}^{Q'H'} + 4\Lambda \Phi_{DB}{}^{Q'}{}_{(E'\epsilon Q')B'} + 4\nabla_{E'(D} \nabla_{B)} \Lambda + 2\phi_{CDBQ} \Phi^C{}_{E'}{}^Q{}_{B'} = 0, \quad (25c)$$

$$\square s - \Xi \Phi_{ACA'C'} \Phi^{ACA'C'} - 3\nabla_{CC'} \Lambda \nabla^{CC'} \Xi = 0, \quad (25d)$$

$$\square \phi_{ABCD} - 6\Xi \phi^{FQ}{}_{(AB} \phi_{CD)FQ} + 12\Lambda \phi_{ABCD} = 0, \quad (25e)$$

$$\square \Xi - 4(s + \Lambda \Xi) = 0, \quad (25f)$$

on \mathcal{U} .

Remark. The unphysical metric is not part of the unknowns of the system of equations of the spinorial version of the CEFÉ. This observation is of relevance in the present context because when the operator \square is applied to a spinor N_K of non-zero range one obtains first derivatives of the connection —if the metric is part of the unknowns then these first derivatives of the connection representing second derivatives of g would enter into the principal part of the operator \square . Therefore, since in this setting the metric is not part of the unknowns, the principal part of the operator \square is given by $\epsilon^{AB} \epsilon^{A'B'} e_{AA'} e_{BB'}$.

Remark. In the sequel let $\{e, \Gamma, \Phi, \phi\}$ denote vector-valued unknowns encoding the independent components of $\{e_{AA'}{}^c, \Gamma_{CC'}{}^{AB}, \Phi_{AA'BB'}, \phi_{ABCD}\}$ and let $\mathbf{u} \equiv (e, \Gamma, \Phi, \phi, s, \Xi)$. Additionally, let $\partial \mathbf{u}$ denote collectively the derivatives of \mathbf{u} . With this notation the wave equations of Proposition 1 can be recast as a quasilinear wave equation for \mathbf{u} having, in local coordinates, the form

$$g^{\mu\nu}(\mathbf{u}) \partial_\mu \partial_\nu \mathbf{u} + \mathbf{F}(x; \mathbf{u}, \partial \mathbf{u}) = 0, \quad (26)$$

where \mathbf{F} is a vector-valued function of its arguments and $g^{\mu\nu}$ denotes the components, in local coordinates, of contravariant version of a Lorentzian metric g . In accordance with our notation $g^{\mu\nu} \equiv \eta^{ab} e_a{}^\mu e_b{}^\nu$ where, in local coordinates, one writes $e_a = e_a{}^\mu \partial_\mu$.

4 Propagation of the constraints and the derivation of the subsidiary system

The starting point of the derivation of the wave equations discussed in the previous section was the CEFE. Therefore, any solution to the CEFE is a solution to the wave equations. It is now natural to ask: under which conditions a solution to the wave equations (25) will imply a solution to the CEFE? The general strategy to answer this question is to use the spinorial wave equations of Proposition 1 to construct a subsidiary system of homogeneous wave equations for the zero-quantities and impose vanishing initial conditions. Then, using a standard existence and uniqueness result, the unique solution satisfying the data will be given by the vanishing of each zero-quantity. This means that under certain conditions (encoded in the initial data for the subsidiary system) a solution to the spinorial wave equations will imply a solution to the original CEFE. The procedure to construct the subsidiary equations for the zero quantities is similar to the construction of the wave equations of Proposition 1. There is, however, a key difference: the covariant derivative is, a priori, not assumed to be a Levi-Civita connection. Instead we assume that the connection is metric but not necessarily torsion-free. We will denote this derivative by $\widehat{\nabla}$. Therefore, whenever a commutator of covariant derivatives appears, or in spinorial terms the operator $\widehat{\square}_{AB} \equiv \widehat{\nabla}_{C'(A} \widehat{\nabla}_{B)}^{C'}$, it is necessary to use the $\widehat{\nabla}$ -spinorial Ricci identities involving a non-vanishing torsion spinor —this generalisation is given in the Appendix A and is required in the discussion of the subsidiary equations where the torsion is, in itself, a variable for which a subsidiary equation needs to be constructed.

As in the previous section, the procedure for obtaining the subsidiary system is similar for each zero-quantity. Therefore, we first give a general outline of the procedure.

4.1 General procedure for obtaining the subsidiary system and the propagation of the constraints

In the general procedure described in Section 3.1, the spinor N_{EAK} played the role of a zero-quantity, while the spinor $M_{AB'\kappa}$ played the role of the variable for which the wave equation (8) was to be derived. In the construction of the subsidiary system we are not interested in finding an equation for $M_{AB'\kappa}$ but in deriving an equation for N_{EAK} under the hypothesis that the wave equation for $M_{AB'\kappa}$ is satisfied. As already discussed, since we cannot assume that the connection is torsion-free the equation for N_{EAK} has to be written in terms of the metric connection $\widehat{\nabla}$.

Before deriving the subsidiary equation let us emphasise an important point. In Section 3.1 we defined $N_{EAK} \equiv \nabla_E^{B'} M_{AB'\kappa}$. Then, decomposing this quantity as usual we obtained

$$N_{EAK} = \nabla_E^{B'} M_{AB'\kappa} + \frac{1}{2} \epsilon_{EA} \nabla^{QB'} M_{QB'\kappa}.$$

At this point in the discussion of Section 3.1 we introduced a gauge source function $\nabla^{PQ'} M_{PQ'\kappa} = F_\kappa$. Now, instead of directly deriving an equation for N_{EAK} we have derived an equation using the modified quantity

$$\widehat{N}_{EAK} \equiv \nabla_E^{B'} M_{AB'\kappa} + \frac{1}{2} \epsilon_{EA} F_\kappa.$$

Accordingly, the wave equations of Proposition 1 can be succinctly written as $\nabla^E_{A'} \widehat{N}_{EAK} = 0$. Later on, we will have to show that, in fact, $\widehat{N}_{EAK} = N_{EAK}$ if the appropriate initial conditions are satisfied. In addition, observe that $\nabla^A_{C'} \widehat{N}_{EAK}$ can be written in terms of the connection $\widehat{\nabla}$ by means of a *transition spinor* $Q_{AA'BC}$ —see Appendix B for the definition. Using equation (65) of Appendix B we get

$$\widehat{\nabla}^A_{C'} \widehat{N}_{ABK} = \nabla^A_{C'} \widehat{N}_{ABK} - Q^A_{C'A}{}^H \widehat{N}_{HBK} - Q^A_{C'BK}{}^H \widehat{N}_{AHK} - \dots - Q^A_{C'K}{}^H \widehat{N}_{AB\dots H} \quad (27)$$

where K is the last index of the string κ . For a connection which is metric, the transition spinor can be written entirely in terms of the torsion as

$$Q_{AA'BC} \equiv -2\Sigma_{BAA'C} - 2\Sigma_{A(C|A'|B)} - 2\bar{\Sigma}_{A'(C|Q'}\epsilon_{A|B)}. \quad (28)$$

If the wave equation $\nabla^A_{C'}\hat{N}_{AB\kappa} = 0$ is satisfied, the first term of equation (27) vanishes. Therefore, the wave equations of Proposition 1 can be written in terms of the connection $\hat{\nabla}$ as

$$\hat{\nabla}^A_{C'}\hat{N}_{AB\kappa} = -Q^A_{C'A}{}^H\hat{N}_{HB\kappa} - Q^A_{C'B}{}^H\hat{N}_{AH\kappa} - \dots - Q^A_{C'K}{}^H\hat{N}_{AB\dots H}. \quad (29)$$

In what follows, the right hand side of the last equation will be denoted by $W_{BC'\kappa}$.

4.1.1 The subsidiary system

Now, we want to show that by setting the appropriated initial conditions, if the wave equation $\nabla^A_{E'}\hat{N}_{AB\kappa} = 0$ holds then $\hat{N}_{AB\kappa} = 0$. The strategy will be to obtain an homogeneous wave equation for $\hat{N}_{AB\kappa}$ written in terms of the connection $\hat{\nabla}$. First, observe that $\hat{\nabla}^{Q'}_P\hat{N}_{AB\kappa}$ can be decomposed as

$$\hat{\nabla}^{Q'}_P\hat{N}_{AB\kappa} = \hat{\nabla}^{Q'}_{(P}\hat{N}_{A)B\kappa} + \frac{1}{2}\epsilon_{PA}\hat{\nabla}^{Q'}_E\hat{N}^E_{B\kappa}. \quad (30)$$

Replacing the second term using (29) —i.e. using that the wave equation $\nabla^A_{E'}\hat{N}_{AB\kappa} = 0$ holds— we get that

$$\hat{\nabla}^{Q'}_P\hat{N}_{AB\kappa} = \hat{\nabla}^{Q'}_{(P}\hat{N}_{A)B\kappa} + \frac{1}{2}\epsilon_{PA}W^{Q'}_{B\kappa}$$

Applying $\hat{\nabla}^P_{Q'}$ to the previous equation and expanding the symmetrised term in the right-hand side one obtains

$$\begin{aligned} \hat{\nabla}^P_{Q'}\hat{\nabla}^{Q'}_P\hat{N}_{AB} &= \frac{1}{2}\hat{\nabla}^P_{Q'}(\hat{\nabla}^{Q'}_P\hat{N}_{AB} + \hat{\nabla}^{Q'}_A\hat{N}_{PB}) + \frac{1}{2}\hat{\nabla}_{AQ'}W^{Q'}_{B\kappa}, \\ &= -\frac{1}{2}\hat{\square}\hat{N}_{AB} - \frac{1}{2}\hat{\nabla}_{PQ'}\hat{\nabla}^{Q'}_A\hat{N}^P_B + \frac{1}{2}\hat{\nabla}_{AQ'}W^{Q'}_{B\kappa}, \\ &= -\frac{1}{2}\hat{\square}\hat{N}_{AB} - \frac{1}{2}(\hat{\square}_{PA}\hat{N}^P_B + \frac{1}{2}\epsilon_{PA}\hat{\square}\hat{N}^P_B) + \frac{1}{2}\hat{\nabla}_{AQ'}W^{Q'}_{B\kappa}. \end{aligned}$$

From this expression, after some rearrangements we obtain

$$\hat{\square}\hat{N}_{AB\kappa} = 2\hat{\square}_{PA}\hat{N}^P_{B\kappa} - 2\hat{\nabla}_{AQ'}W^{Q'}_{B\kappa}.$$

It only remains to reexpress the right-hand side of the above equation using the $\hat{\nabla}$ - spinorial Ricci identities. This can be computed for each zero-quantity using the expressions given in Appendix A. Observe that the result is always an homogeneous expression in the zero-quantities and its first derivatives. The last term also shares this property since the transition spinor can be completely written in terms of the torsion, as shown in equation (28), which is one of the zero-quantities. Finally, once the homogeneous wave equation is obtained we set the initial conditions

$$\hat{N}_{AB\kappa}|_{\mathcal{S}} = 0 \quad \text{and} \quad (\hat{\nabla}_{EE'}\hat{N}_{AB\kappa})|_{\mathcal{S}} = 0$$

on a initial hypersurface \mathcal{S} , and using a standard result of existence and uniqueness for wave equations we conclude that the unique solution satisfying this data is $\hat{N}_{AB\kappa} = 0$.

Remark. The crucial step in the last derivation was the assumption that the equation $\nabla^A_{E'}\hat{N}_{AB\kappa} = 0$ is satisfied —i.e. the wave equation (8) for $M_{AB\kappa}$.

4.1.2 Initial data for the subsidiary system

Now, we take a closer look at the initial conditions

$$\widehat{N}_{AB\kappa}|_S = 0, \quad (\widehat{\nabla}_{EE'}\widehat{N}_{AB\kappa})|_S = 0.$$

As will be shown in the sequel, these conditions will be used to construct initial data for the wave equations of Proposition 1. The important observation is that only $\widehat{N}_{AB\kappa}|_S = 0$ is essential, while $\widehat{\nabla}_{EE'}\widehat{N}_{AB}|_S = 0$ holds by virtue of the condition $\nabla^{AA'}\widehat{N}_{AB\kappa} = 0$. In order to show this, first observe that as the spatial derivatives of $\widehat{N}_{AB\kappa}$ can be determined from $\widehat{N}_{AB\kappa}|_S = 0$, it follows that $(\widehat{\nabla}_{EE'}\widehat{N}_{AB\kappa})|_S = 0$ is equivalent to only specify the derivative along the normal to S .

Let $\tau^{AA'}$ be an Hermitian spinor corresponding to a timelike vector such that $\tau^{AA'}|_S$ is the normal to S . The spinor $\tau^{AA'}$ can be used to perform a *space spinor split* of the derivative $\widehat{\nabla}_{AA'}$:

$$\widehat{\nabla}_{AA'} = \frac{1}{2}\tau_{AA'}\mathcal{P} - \tau_{A'}^Q\mathcal{D}_{AQ}$$

where

$$\mathcal{P} \equiv \tau^{AA'}\widehat{\nabla}_{AA'} \quad \text{and} \quad \mathcal{D}_{AB} \equiv \tau_{(B}{}^{A'}\widehat{\nabla}_{A)A'}$$

denote, respectively, the derivative along the direction given by $\tau^{AA'}$ and \mathcal{D}_{AB} is the *Sen connection* relative to $\tau^{AA'}$ ². We have chosen the normalisation $\tau^{AA'}\tau_{AA'} = 2$, in accordance with the conventions of [11]. Using this split and $\widehat{N}_{AB\kappa}|_S = 0$ it follows that

$$\widehat{\nabla}_{EE'}\widehat{N}_{AB\kappa}|_S = \frac{1}{2}(\tau_{EE'}\mathcal{P}\widehat{N}_{AB\kappa})|_S.$$

Therefore, requiring $\widehat{\nabla}_{EE'}\widehat{N}_{AB}|_S = 0$ is equivalent to having $(\mathcal{P}\widehat{N}_{AB\kappa})|_S = 0$ as previously stated. Now, observe that the wave equation $\nabla^{AA'}\widehat{N}_{AB\kappa} = 0$ or, equivalently, $\widehat{\nabla}^{AA'}\widehat{N}_{AB\kappa} = W^{A'}{}_{B\kappa}$ implies $(\widehat{\nabla}^{AA'}\widehat{N}_{AB\kappa})|_S = W^{A'}{}_{B\kappa}|_S$ ³. Therefore, if we require that all the zero-quantities vanish on the initial hypersurface S then $(\widehat{\nabla}^{AA'}\widehat{N}_{AB\kappa})|_S = 0$. Using, again, the space spinor decomposition of $\widehat{\nabla}_{AA'}$ and considering $\widehat{N}_{AB\kappa}|_S = 0$ we get $(\tau^{AA'}\mathcal{P}\widehat{N}_{AB\kappa})|_S = 0$ which also implies that $(\mathcal{P}\widehat{N}_{AB\kappa})|_S = 0$.

Summarising, the only condition that is needed is that all the zero-quantities vanish on the initial hypersurface S since the condition $(\widehat{\nabla}_{EE'}\widehat{N}_{AB\kappa})|_S = 0$ is always satisfied by virtue of the wave equation $\nabla^{AA'}\widehat{N}_{AB\kappa} = 0$.

4.1.3 Propagation of the constraints

We still need to show that $\widehat{N}_{AB} = N_{AB}$. One can write

$$N_{AB\kappa} - \widehat{N}_{AB\kappa} = \frac{1}{2}\epsilon_{AB}Q_\kappa,$$

where Q_κ encodes the difference between $\widehat{N}_{AB\kappa}$ and $N_{AB\kappa}$. Computing the trace of the last equation and taking into account the definition of $N_{AB\kappa}$ one finds that $\widehat{N}^A{}_{A\kappa} = Q_\kappa$. Now, invoking the results derived in the last subsection it follows that if the wave equation $\nabla^A{}_{E'}\widehat{N}_{AB\kappa} = 0$ is satisfied and all the zero-quantities vanish on the initial hypersurface S then $\widehat{N}_{AB\kappa} = 0$. This observation also implies that if $\widehat{N}^A{}_{A\kappa}|_S = 0$ then $\widehat{N}^A{}_{A\kappa} = 0$. The later result, expressed in terms of Q_κ means that if $Q_\kappa|_S = 0$ then $Q_\kappa = 0$. Therefore, requiring that all the zero-quantities

²Under certain conditions the Sen connection coincides with the Levi-Civita connection of the intrinsic 3-metric of the hypersurfaces orthogonal to $\tau^{AA'}$.

³Recall that $W^{A'}{}_{B\kappa}|_S$ is given entirely in terms of zero-quantities since the transition spinor can be written in terms of the torsion.

vanish on \mathcal{S} and that the wave equation $\nabla^{AA'} \widehat{N}_{AB\kappa} = 0$ holds everywhere, is enough to ensure that

$$\widehat{N}_{AB\kappa} = N_{AB\kappa}$$

everywhere. Moreover, $\widehat{N}_{AB\kappa} = 0$ implies that $N_{AB\kappa} = 0$ and the gauge conditions hold. Namely, one has that

$$\nabla^{AB'} M_{AB'\kappa} = F_\kappa(x).$$

4.2 Subsidiary system and propagation of the constraints

The essential ideas of the Section 4.1 can be applied to every single zero-quantity. One only needs to take into account the particular index structure of each zero-quantity encoded in the string of spinor indices κ . The problem then reduces to the computation of

$$\widehat{\square}_{PA} \widehat{N}^P{}_{B\kappa}, \quad \widehat{\nabla}_{AQ'} W^{Q'}{}_{B\kappa},$$

the result of which is to be substituted into

$$\widehat{\square} \widehat{N}_{AB\kappa} = 2\widehat{\square}_{PA} \widehat{N}^P{}_{B\kappa} - 2\widehat{\nabla}_{AQ'} W^{Q'}{}_{B\kappa}. \quad (31)$$

The latter can be succinctly computed using the equations (61a) in Appendix A. The explicit form can be easily obtained and renders long expressions for each zero-quantity. The key observation from these computations is that (31) leads to an homogeneous wave equation. The explicit form is given in Appendix C. These results can be summarised in the following proposition:

Proposition 2. *Assume that the wave equations*

$$\begin{aligned} \nabla^A{}_{E'} \widehat{\Sigma}_{AB}{}^c &= 0, & \nabla^{C'}{}_D \widehat{\Xi}_{ABC'D'} &= 0, \\ \nabla^C{}_{E'} \widehat{\Delta}_{CDBB'} &= 0, & \nabla_E{}^{B'} \Lambda_{BB'AC} &= 0, \\ \nabla_{AA'} Z^{AA'} &= 0, & Z_{AA'}{}^{AA'} &= 0, \end{aligned}$$

are satisfied everywhere. Then the zero-quantities satisfy the homogeneous wave equations

$$\begin{aligned} \widehat{\square} \widehat{\Sigma}_{AB}{}^c - 2\widehat{\square}_{PA} \widehat{\Sigma}^P{}_{B}{}^c + 2\widehat{\nabla}_{AQ'} W[\Sigma]^{Q'}{}_{B}{}^c &= 0, \\ \widehat{\square} \widehat{\Xi}_{ABC'D'} - 2\widehat{\square}_{P'C'} \widehat{\Xi}_{AB}{}^{P'}{}_{D'} + 2\widehat{\nabla}_{C'Q} W[\Xi]^{Q'}{}_{ABD'} &= 0, \\ \widehat{\square} \widehat{\Delta}^P{}_{DBB'} - 2\widehat{\square}_{PC} \widehat{\Delta}^P{}_{DBB'} + 2\widehat{\nabla}_{CQ'} W[\Delta]^{Q'}{}_{DBB'} &= 0, \\ \widehat{\square} \Lambda_{BB'AC} - 2\widehat{\square}_{P'B'} \Lambda_B{}^{P'}{}_{AC} + 2\widehat{\nabla}_{B'Q} W[\Lambda]^Q{}_{BAC} &= 0, \\ \widehat{\square} Z_{AA'}{}^{AA'} - W[Z]^{AA'}{}_{AA'} &= 0, \end{aligned}$$

where

$$\begin{aligned} W[\Sigma]^{Q'}{}_{B}{}^c &\equiv \widehat{\nabla}^{Q'}{}_E \widehat{\Sigma}^E{}_{B}{}^c, & W[\Xi]^Q{}_{ABD'} &\equiv \widehat{\nabla}^Q{}_{E'} \widehat{\Xi}_{AB}{}^{E'}{}_{D'}, & W[\Delta]^{Q'}{}_{DBB'} &\equiv \widehat{\nabla}^{Q'}{}_F \widehat{\Delta}^F{}_{DBB'}, \\ W[\Lambda]^Q{}_{BAC} &\equiv \widehat{\nabla}^{Q'}{}_{E'} \Lambda_B{}^{E'}{}_{AC}, & W[Z]^{AA'}{}_{AA'} &\equiv \widehat{\nabla}^{AA'} Z_{AA'}. \end{aligned}$$

We will refer to the set of equations given in the last proposition as the *subsidiary system*. It should be noticed that the terms of the form $\widehat{\square}_{PA} \widehat{N}^P{}_{B\kappa}$ and $W^{Q'}{}_{B\kappa}$ can be computed using the $\widehat{\nabla}$ -Ricci identities and the transition spinor $Q_{AA'BC}$ respectively. Using the subsidiary equations from the previous proposition one readily obtains the following *Reduction Lemma*:

Proposition 3. *If the initial data for the subsidiary system of Proposition 2 is given by*

$$\widehat{\Sigma}_{AB}{}^c|_{\mathcal{S}} = 0, \quad \widehat{\Xi}_{ABC'D'}|_{\mathcal{S}} = 0, \quad \widehat{\Delta}_{ABCC'}|_{\mathcal{S}} = 0, \quad \Lambda_{BB'AC}|_{\mathcal{S}} = 0, \quad Z_{AA'}|_{\mathcal{S}} = 0,$$

where \mathcal{S} is an spacelike hypersurface, and the wave equations of Proposition 1 are satisfied everywhere, then one has a solution to the vacuum CEFE —in other words

$$\Sigma_{AB}{}^c = 0, \quad \Xi_{ABC'D'} = 0, \quad \Delta_{ABCC'} = 0, \quad \Lambda_{BB'AC} = 0, \quad Z_{AA'} = 0,$$

in $D(\mathcal{S})$. Moreover, whenever $\Xi \neq 0$, the solution to the CEFE implies a solution to the vacuum Einstein field equations.

Proof. It can be verified, using the $\widehat{\nabla}$ -Ricci identities given in the Appendix A, that the equations of Proposition 2 are homogeneous wave equations for the zero-quantities. Notice, however, that the equation for $Z_{AA'}$ is not a wave equation but of first order and homogeneous. Therefore, if we impose that the zero-quantities vanish on an initial spacelike hypersurface \mathcal{S} then by the homogeneity of the equations we have that

$$\widehat{\Sigma}_{AB}{}^c = 0, \quad \widehat{\Xi}_{ABC'D'} = 0, \quad \widehat{\Delta}_{ABCC'} = 0, \quad \Lambda_{BB'AC} = 0, \quad Z_{AA'} = 0,$$

everywhere on $D(\mathcal{S})$. Moreover, since initially $\widehat{\Sigma}_{AB}{}^c = \Sigma_{AB}{}^c$, $\widehat{\Xi}_{ABC'D'} = \Xi_{ABC'D'}$ and $\widehat{\Delta}_{ABCC'} = \Delta_{ABCC'}$, we have that $\Sigma_{AB}{}^c = 0$, $\Xi_{ABC'D'} = 0$, $\Delta_{ABCC'} = 0$ on $D(\mathcal{S})$. In addition, using that a solution to the CEFE implies a solution to the Einstein field equations whenever $\Xi \neq 0$ [8], it follows that a solution to the wave equations of Proposition 1 with initial data consistent with the initial conditions given in Proposition 3 will imply a solution to the vacuum Einstein field equations whenever $\Xi \neq 0$. \square

Remark. It is noticed that the initial data for the subsidiary equations gives a way to specify the data for the wave equations of Proposition 1. This observation is readily implemented through a space spinor formalism which mimics the hyperbolic reduction process to extract a first order hyperbolic system out of the CEFE —see e.g. [11]. In order to illustrate this procedure let us consider the data for the rescaled Weyl spinor encoded in $\Lambda_{A'BCD}|_{\mathcal{S}} = 0$.

4.2.1 Initial data for the wave equations (rescaled Weyl spinor)

We need to provide the initial data

$$\phi_{ABCD}|_{\mathcal{S}}, \quad \mathcal{P}\phi_{ABCD}|_{\mathcal{S}}.$$

A convenient way to specify the initial data is to use the space spinor formalism to split the equations encoded in $\Lambda_{A'BCD} = 0$. From this split, a system of evolution and constraint equations can be obtained. Recall that $\Lambda_{A'BCD} \equiv \nabla^Q{}_{A'}\phi_{ABCQ}$. Making use of the decomposition of $\nabla_{AB} \equiv \tau_B{}^{A'}\nabla_{AA'}$ in terms of the operators \mathcal{P} and \mathcal{D}_{AB} we get

$$\Lambda_{ABCD} = -\frac{1}{2}\mathcal{P}\phi_{ABCD} + \mathcal{D}^Q{}_A\phi_{BCDQ}$$

Evolution and constraint equations are obtained, respectively, from considering

$$E_{ABCD} \equiv -2\Lambda_{ABCD} = \mathcal{P}\phi_{ABCD} - 2\mathcal{D}^Q{}_A\phi_{BCDQ} = 0, \quad (\text{evolution equation})$$

$$C_{CD} \equiv \Lambda^Q{}_{QCD} = \mathcal{D}^P{}_Q\phi_{PQCD} = 0 \quad (\text{constraint equation}).$$

Restricting the last equations to the initial hypersurface \mathcal{S} it follows that the initial data $\phi_{PQCD}|_{\mathcal{S}}$ must satisfy $C_{CD}|_{\mathcal{S}} = 0$ and the initial data for $(\mathcal{P}\phi_{PQCD})|_{\mathcal{S}}$ can be read from $E_{ABCD}|_{\mathcal{S}} = 0$.

The procedure for the other equations is analogous and can be succinctly obtained by revisiting the derivation of the first order hyperbolic equations derived from the CEFE using the space spinor formalism —see for instance [11].

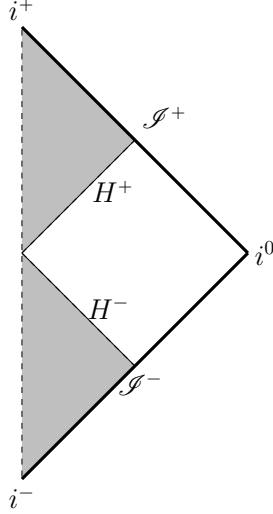


Figure 1: Penrose Diagram for the Milne Universe. The Milne Universe diagram correspond to the non-spatial (shaded area) portion of the Penrose diagram of the Minkowski spacetime. The boundary $H^+ \cup H^-$ corresponds to the limit of the region where the coordinates (t, χ) are well defined.

5 Analysis of the Milne Universe

As an application of the hyperbolic reduction procedure described in the previous sections we analyse the stability of the *Milne Universe*, $(\tilde{\mathcal{M}}, \tilde{\mathbf{g}})$. This spacetime is a Friedman-Lemaître-Robinson-Walker vacuum solution with vanishing Cosmological constant, energy density and pressure. In fact, it represents flat space written in comoving coordinates of the world-lines starting at $t = 0$ —see [13]. This means that the Milne Universe can be seen as a portion of the Minkowski spacetime, which we know is conformally related to the *Einstein Cosmos*, $(\mathcal{M}_E \equiv \mathbb{R} \times \mathbb{S}^3, \mathring{\mathbf{g}})$ (sometimes also called the *Einstein cylinder*) —see Figure 1. The metric $\tilde{\mathbf{g}}$ of the Milne Universe is given in comoving coordinates $(t, \chi, \theta, \varphi)$ by

$$\tilde{\mathbf{g}} = dt \otimes dt - t^2 (d\chi \otimes d\chi + \sin^2 \chi (d\theta \otimes d\theta + \sin^2 \theta d\varphi \otimes d\varphi)) \quad (32)$$

where

$$t \in (-\infty, \infty), \quad \chi \in [0, \infty), \quad \theta \in [0, \pi], \quad \phi \in [0, 2\pi).$$

In fact, introducing the coordinates

$$\bar{r} \equiv t \sinh \chi, \quad \bar{t} \equiv t \cosh \chi$$

the metric reads

$$\tilde{\mathbf{g}} = d\bar{t} \otimes d\bar{t} - d\bar{r} \otimes d\bar{r} - \bar{r}^2 (d\theta \otimes d\theta + \sin^2 \theta d\varphi \otimes d\varphi).$$

Therefore $\bar{t}^2 - \bar{r}^2 > 0$, and the Milne Universe corresponds to the non-spatial region of Minkowski spacetime as shown in the Penrose diagram of Figure 1. As already discussed, this metric is conformally related to the metric $\mathring{\mathbf{g}}$ of the Einstein Cosmos. More precisely, one has that

$$\mathring{\mathbf{g}} = \Xi^2 \tilde{\mathbf{g}}$$

where the metric of the Einstein cylinder, $\mathring{\mathbf{g}}$, is given by

$$\mathring{\mathbf{g}} \equiv dT \otimes dT - \mathbf{h},$$

with \mathbf{h} denoting the standard metric of \mathbb{S}^3

$$\mathbf{h} \equiv \mathbf{d}\psi \otimes \mathbf{d}\psi + \sin^2 \psi \mathbf{d}\theta \otimes \mathbf{d}\theta + \sin^2 \psi \sin^2 \theta \mathbf{d}\varphi \otimes \mathbf{d}\varphi.$$

The conformal factor relating the metric of the Milne Universe to metric of the Einstein Universe is given by

$$\overset{\circ}{\Xi} = \cos T + \cos \psi,$$

and the coordinates (T, ψ) are related to (\bar{t}, \bar{r}) via

$$T = \arctan(\bar{t} + \bar{r}) + \arctan(\bar{t} - \bar{r}), \quad \psi = \arctan(\bar{t} + \bar{r}) - \arctan(\bar{t} - \bar{r}).$$

Equivalently, in terms of the original coordinates t and χ we have

$$\chi = \arctan\left(\frac{\sin \psi}{\sin T}\right), \quad t = \sqrt{\frac{\cos \psi - \cos T}{\cos \psi + \cos T}}.$$

Therefore, the Milne Universe is conformal to the domain

$$\tilde{\mathcal{M}} = \{p \in \mathcal{M}_E \mid 0 \leq \psi < \pi, \quad \psi - \pi < T < \pi - \psi, \quad |T| > \psi\}.$$

5.1 The Milne Universe as a solution to the wave equations

Since the Milne Universe is a solution to the the Einstein field equations, it follows that the pair $(\overset{\circ}{g}, \overset{\circ}{\Xi})$ implies a solution to the CEFÉ which, in turn, constitutes a solution to the wave equations of Proposition 1. Following the discussion of Section 2, this solution consists of the frame fields

$$\{e_a{}^c, \Gamma_a{}^b{}_c, L_{ab}, d^a{}_{bcd}, \Sigma_a, \Xi, s\}$$

or, equivalently, the spinorial fields

$$\{e_{AA'}{}^c, \Gamma_{AA'}{}^B{}_C, \Phi_{AA'BB'}, \phi_{ABCD}, \Sigma_{AA'}, \Sigma, \Xi, s\}$$

where we have written $\Sigma_a \equiv \nabla_a \Xi$ and $\nabla_{AA'} \Xi \equiv \Sigma_{AA'}$ as a shorthand for the derivative of the conformal factor.

For later use, we notice that in the Einstein Cosmos $(\mathcal{M}_E, \overset{\circ}{g})$ we have

$$\mathbf{Weyl}[\overset{\circ}{g}] = 0, \quad \mathbf{R}[\overset{\circ}{g}] = -6, \quad \mathbf{Schouten}[\overset{\circ}{g}] = \frac{1}{2}(\mathbf{d}T \otimes \mathbf{d}T + \mathbf{h}).$$

The spinorial version of the above tensors can be more easily expressed in terms of a frame. To this end, now consider a geodesic on the Einstein Cosmos $(\mathcal{M}_E, \overset{\circ}{g})$ given by

$$x(\tau) = (\tau, x_\star), \quad \tau \in \mathbb{R},$$

where $x_\star \in \mathbb{S}^3$ is fixed. Using the congruence of geodesics generated varying x_\star over \mathbb{S}^3 we obtain a Gaussian system of coordinates (τ, x^α) on the Einstein cylinder $\mathbb{R} \times \mathbb{S}^3$ where (x^α) are some local coordinates on \mathbb{S}^3 . In addition, in a slight abuse of notation *we identify the standard time coordinate T on the Einstein cylinder with the parameter τ of the geodesic.*

5.1.1 Frame expressions

A globally defined orthonormal frame on the Einstein Cosmos $(\mathcal{M}_E, \overset{\circ}{g})$ can be constructed by first considering the linearly independent vector fields in \mathbb{R}^4

$$\begin{aligned} \mathbf{c}_1 &\equiv w \frac{\partial}{\partial z} - z \frac{\partial}{\partial w} + x \frac{\partial}{\partial y} - y \frac{\partial}{\partial x}, \\ \mathbf{c}_2 &\equiv w \frac{\partial}{\partial y} - y \frac{\partial}{\partial w} + z \frac{\partial}{\partial x} - x \frac{\partial}{\partial z}, \\ \mathbf{c}_3 &\equiv w \frac{\partial}{\partial x} - x \frac{\partial}{\partial w} + y \frac{\partial}{\partial z} - z \frac{\partial}{\partial y}, \end{aligned}$$

where (w, x, y, z) are Cartesian coordinates in \mathbb{R}^4 . The vectors $\{\mathbf{c}_i\}$ are tangent to \mathbb{S}^3 and form a global frame for \mathbb{S}^3 —see e.g. [23]. This spatial frame can be extended to a spacetime frame $\{\mathring{\mathbf{e}}_a\}$ by setting $\mathring{\mathbf{e}}_0 \equiv \partial_\tau$ and $\mathring{\mathbf{e}}_i \equiv \mathbf{c}_i$. Using this notation we observe that the components of the basis respect to this frame are given by $\mathring{\mathbf{e}}_a = \delta_a^b \mathbf{c}_b \equiv \mathring{\mathbf{e}}_a^b \mathbf{c}_b$. With respect to this orthogonal basis the components of the Schouten tensor are given by

$$\mathring{L}_{ab} = \delta_a^0 \delta_b^0 - \frac{1}{2} \eta_{ab}.$$

so that the components of the traceless Ricci tensor are given by

$$\mathring{R}_{\{ab\}} = 2\delta_a^0 \delta_b^0 - \frac{1}{2} \eta_{ab}$$

where the curly bracket around the indices denote the symmetric trace-free part of the tensor. In addition,

$$\mathring{d}_{abcd} = 0$$

since the Weyl tensor vanishes.

Now, let $\mathring{\gamma}_i^j \mathbf{k}$ denote the connection coefficients of the Levi-Civita connection \mathbf{D} of \mathbf{h} with respect to the spatial frame $\{\mathbf{c}_i\}$. Observe that the structure coefficients defined by $[\mathbf{c}_i, \mathbf{c}_j] = C_i^k \mathbf{c}_k$ are given by

$$\mathring{\gamma}_i^k \mathbf{k}_j = -\epsilon_i^k \mathbf{k}_j$$

where $\epsilon_i^k \mathbf{k}_j$ is the 3-dimensional Levi-Civita totally antisymmetric tensor. Taking into account that $\mathring{\mathbf{e}}_0 = \partial_\tau$ is a timelike Killing vector of $\mathring{\mathbf{g}}$, we can readily obtain the connection coefficients $\mathring{\Gamma}_a^b \mathbf{c}$, of the Levi-Civita connection $\mathring{\nabla}$ of the metric $\mathring{\mathbf{g}}$, with respect to the basis $\{\mathring{\mathbf{e}}_a\}$. More precisely, one has that

$$\mathring{\Gamma}_a^b \mathbf{c} = -\epsilon_0^b \mathbf{c}_a.$$

For the conformal factor and its concomitants we readily obtain

$$\mathring{\Sigma} \equiv \mathring{\Sigma}_0 = -\sin \tau, \quad \mathring{\Sigma}_3 = -\sin \psi, \quad \mathring{\Sigma}_1 = \mathring{\Sigma}_2 = 0, \quad \mathring{s} = -\frac{1}{2}(\cos \tau + \cos \psi).$$

5.1.2 Spinorial expressions

In order to obtain the spinor frame form of the last expressions let $\tau^{\mathbf{A}\mathbf{A}'}$ denote the spinorial counterpart of the vector $\sqrt{2}\partial_\tau$ so that $\tau_{\mathbf{A}\mathbf{A}'}\tau^{\mathbf{A}\mathbf{A}'} = 2$. With this choice, consider a spinor dyad $\{\epsilon_{\mathbf{A}}^A\} = \{o^A, \iota^A\}$ adapted to $\tau^{\mathbf{A}\mathbf{A}'}$ —i.e. a spinor dyad such that

$$\tau_{\mathbf{A}\mathbf{A}'} = \epsilon_0^A \epsilon_{0'}^{A'} + \epsilon_1^A \epsilon_{1'}^{A'}. \quad (33)$$

The spinor $\tau^{\mathbf{A}\mathbf{A}'}$ can be used to introduce a space spinor formalism similar to the one discussed in Section 4.1.2. Now, the *Infeld-van der Waerden symbols* $\sigma_{\mathbf{A}\mathbf{A}'}^b$ are given by the matrices

$$\begin{aligned} \sigma_{\mathbf{A}\mathbf{A}'}^0 &\equiv \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, & \sigma_{\mathbf{A}\mathbf{A}'}^1 &\equiv \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \\ \sigma_{\mathbf{A}\mathbf{A}'}^2 &\equiv \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, & \sigma_{\mathbf{A}\mathbf{A}'}^3 &\equiv \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \end{aligned}$$

One directly finds that in the present case

$$\mathring{\mathbf{e}}_{\mathbf{A}\mathbf{A}'}^b \equiv \sigma_{\mathbf{A}\mathbf{A}'}^a \mathring{\mathbf{e}}_a^b = \sigma_{\mathbf{A}\mathbf{A}'}^b. \quad (34)$$

Now, decomposing $\mathring{\mathbf{e}}_{\mathbf{A}\mathbf{A}'}^b$ using the space spinor formalism induced by $\tau^{\mathbf{A}\mathbf{A}'}$ one has that

$$\mathring{\mathbf{e}}_{\mathbf{A}\mathbf{A}'}^a = \frac{1}{2} \tau_{\mathbf{A}\mathbf{A}'} \mathring{\mathbf{e}}^a - \tau_{\mathbf{A}'}^Q \mathring{\mathbf{e}}_{\mathbf{A}Q}^a$$

where

$$\dot{e}^a \equiv \dot{e}_{AA'}^a \tau^{AA'}, \quad \dot{e}_{AB}^a \equiv \tau_{(A}^{Q'} \dot{e}_{B)Q'}^a.$$

Comparing the above expression with equation (34) one readily finds that the coefficients \dot{e}^a and \dot{e}_{AB}^a are given by

$$\begin{aligned} \dot{e}^0 &= \sqrt{2}, & \dot{e}_{AB}^0 &= 0, \\ \dot{e}^i &= 0, & \dot{e}_{AB}^i &= \sigma_{AB}^i. \end{aligned}$$

where σ_{AB}^i are the spatial Infeld-van der Waerden symbols, given by the matrices

$$\sigma_{AB}^1 \equiv \frac{1}{\sqrt{2}} \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \sigma_{AB}^2 \equiv \frac{1}{\sqrt{2}} \begin{pmatrix} i & 0 \\ 0 & i \end{pmatrix}, \quad \sigma_{AB}^3 \equiv \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

The above expressions provide a direct way of recasting the frame expressions of Section 5.1.1 in spinorial terms.

Now, denoting by $2\dot{\Phi}_{AA'BB'}$ the spinorial counterpart of $\dot{R}_{\{ab\}}$ we get

$$\dot{\Phi}_{AA'BB'} = \frac{1}{2} \sigma_{AA'}^a \sigma_{BB'}^b \dot{R}_{\{ab\}} = \sigma_{AA'}^0 \sigma_{BB'}^0 - \frac{1}{4} \epsilon_{AB} \epsilon_{A'B'}.$$

From (33) we see that $\tau_{AA'} = \sqrt{2} \sigma_{AA'}^0$. Accordingly,

$$\dot{\Phi}_{AA'BB'} = \frac{1}{2} \tau_{AA'} \tau_{BB'} - \frac{1}{4} \epsilon_{AB} \epsilon_{A'B'}.$$

To obtain the reduced spin connection coefficients we proceed as follows: let $\dot{\Gamma}_{AA'}^{BB'}{}_{CC'}$ denote the spinorial counterpart of $\dot{\Gamma}_a^b{}_c$. Now, since $\dot{\Gamma}_a^b{}_c = -\epsilon_{0a}^b{}_c$, we can readily compute its spinorial counterpart by recalling the spinorial version of the volume form

$$\epsilon_{AA'BB'CC'DD'} = i(\epsilon_{AC}\epsilon_{BD}\epsilon_{A'D'}\epsilon_{B'C'} - \epsilon_{AD}\epsilon_{BC}\epsilon_{A'C'}\epsilon_{B'D'}).$$

It follows then that

$$\dot{\Gamma}_{BB'}^{CC'}{}_{DD'} = -\frac{1}{\sqrt{2}} \tau^{AA'} \epsilon_{AA'BB'}{}^{CC'}{}_{DD'} = -\frac{i}{\sqrt{2}} (\tau^C{}_{D'} \epsilon_{BD} \epsilon_{B'}{}^{C'} - \tau_D{}^{C'} \epsilon_B{}^C \epsilon_{B'D'})$$

Combining the last expression with the definition of the reduced spin connection coefficients $\dot{\Gamma}_{AA'}^C{}_B$ given in equation (2) one obtains

$$\dot{\Gamma}_{BB'}^C{}_D = -\frac{i}{2\sqrt{2}} (\tau^C{}_{Q'} \epsilon_{BD} \delta_{B'}{}^{Q'} - \tau_D{}^{Q'} \epsilon_{B'Q'} \delta_B{}^C) = -\frac{i}{2\sqrt{2}} (\epsilon_{BD} \tau^C{}_{B'} + \tau_{DB'} \delta_B{}^C)$$

Thus, one concludes that

$$\dot{\Gamma}_{AA'BC} = -\frac{i}{\sqrt{2}} \epsilon_A(\epsilon_C \tau_B)_{A'}$$

Finally, for the rescaled Weyl tensor we simply have

$$\dot{\phi}_{ABCD} = 0.$$

5.1.3 Gauge source functions for the Milne spacetime

The expressions for $\dot{\Gamma}_a^b{}_c$ and \dot{e}_b^a derived in the previous sections allow to readily compute the gauge source functions associated to the conformal representation of the Milne Universe under consideration. Regarding \dot{e}_b^a as the component of a contravariant tensor we compute

$$\dot{\nabla}^b \dot{e}_b^a = \eta^{cb} \dot{\nabla}_c \dot{e}_b^a = \eta^{cb} (\dot{e}_c(\dot{e}_b^a) - \dot{\Gamma}_c^e{}_b \dot{e}_e^a)$$

where $\dot{e}_c = \dot{e}_c^e c_e$. Using that for this case $\dot{e}_b^a = \delta_b^a$, we get

$$\dot{\nabla}^b \dot{e}_b^a = -\eta^{cb} \dot{\Gamma}_c^a{}_b = \eta^{cb} \epsilon_{0c}^a{}_b = 0.$$

Therefore, the coordinate gauge source function vanishes. That is, one has that

$$\mathring{F}^a(x) = \mathring{\nabla}^{AA'} \mathring{e}_{AA'}^a = 0.$$

We proceed in similar way to compute the frame gauge source function. One has that

$$\begin{aligned} \mathring{\nabla}^a \mathring{\Gamma}_a^b{}_c &= \eta^{da} \mathring{\nabla}_d \mathring{\Gamma}_a^b{}_c = \eta^{da} (\mathring{e}_d(\mathring{\Gamma}_a^b{}_c) + \mathring{\Gamma}_d^b{}_e \mathring{\Gamma}_a^e{}_c - \mathring{\Gamma}_d^e{}_a \mathring{\Gamma}_e^b{}_c - \mathring{\Gamma}_d^e{}_c \mathring{\Gamma}_e^b{}_a) \\ &= -\eta^{da} \mathring{e}_d(\epsilon_{0a}^d{}_c) + \eta^{da} \epsilon_{0d}^b{}_e \epsilon_{0a}^e{}_c - \eta^{da} \epsilon_{0d}^e{}_a \epsilon_{0e}^b{}_c - \eta^{da} \epsilon_{0d}^e{}_c \epsilon_{0e}^b{}_a \\ &= \epsilon_0^{ab} \epsilon_{0a}^e{}_c - \epsilon_{0a}^b \epsilon_0^{ae}{}_c = 0. \end{aligned}$$

Therefore, using the irreducible decomposition of $\mathring{\Gamma}_{AA'}^{BB'}{}_{CC'}$ in terms of $\mathring{\Gamma}_{AA'}^B{}_C$ given in (2), we conclude that

$$\mathring{F}_{AB}(x) = \nabla^{QQ'} \mathring{\Gamma}_{QQ'}{}_{AB} = 0.$$

Finally, the conformal gauge source function is determined by the value of the Ricci scalar, in this case $R = -6$. It follows then that

$$\Lambda = \frac{1}{4}.$$

5.1.4 Summary

We collect the main results in the following proposition:

Proposition 4. *The fields $(\mathring{\Xi}, \mathring{\Sigma}, \mathring{\Sigma}_i, \mathring{s}, \mathring{e}_a^b, \mathring{\Gamma}_a^b{}_c, \mathring{L}_{ab}, \mathring{d}^a{}_{bcd})$ given by*

$$\begin{aligned} \mathring{\Xi} &= \cos \tau + \cos \psi, & \mathring{\Sigma} &= -\sin \tau, & \mathring{\Sigma}_3 &= -\sin \psi, & \mathring{e}_a^b &= \delta_a^b, \\ \mathring{\Gamma}_a^b{}_c &= -\epsilon_{0a}^b{}_c, & \mathring{L}_{ab} &= 2\delta_a^0 \delta_b^0 - \frac{1}{2}\eta_{ab}, & \mathring{d}^a{}_{bcd} &= 0, & \mathring{s} &= -\frac{1}{2}(\cos \tau + \cos \psi), \end{aligned}$$

or, alternatively, in spinorial terms, the fields $(\mathring{\Xi}, \mathring{\Sigma}, \mathring{\Sigma}_{AA'}, \mathring{s}, \mathring{e}_{AA'}^b, \mathring{\Gamma}_{AA'}^B{}_C, \mathring{\Phi}_{AA'BB'}, \mathring{\phi}_{ABCD})$ with $\mathring{\Xi}, \mathring{\Sigma}, \mathring{s}$ as above and

$$\begin{aligned} \mathring{e}_{AA'}^b &= \sigma_{AA'}^b, & \mathring{\Gamma}_{AA'}^B{}_C &= -\frac{i}{\sqrt{2}}\epsilon_{A(B}\tau_{C)A'}, & \mathring{\Phi}_{AA'BB'} &= \frac{1}{2}\tau_{AA'}\tau_{BB'} - \frac{1}{4}\epsilon_{AB}\epsilon_{AC}, \\ \mathring{\phi}_{ABCD} &= 0 & \Sigma_{AA'} &= -\sin \psi \sigma_{AA'}^3, \end{aligned}$$

defined on the Einstein cylinder $\mathbb{R} \times \mathbb{S}^3$ constitute a solution to the conformal Einstein field equations representing the Milne Universe. The gauge source functions associated to this representation are given by

$$\mathring{F}^a(x) = 0, \quad \mathring{F}_{AB}(x) = 0, \quad \mathring{\Lambda} = \frac{1}{4}.$$

5.2 Perturbation of initial data

In order to discuss the stability of the Milne Universe using the conformal wave equations we need to find a way to parametrise perturbations of initial data close to the data for the exact solution.

A *basic initial data set for the conformal field equations* consists of a collection $(\mathcal{S}, \mathbf{h}, \mathbf{K}, \Omega, \Sigma)$ such that \mathcal{S} denotes a 3-dimensional manifold, \mathbf{h} is a 3-metric, \mathbf{K} is a symmetric tensor, Ω and Σ are scalar functions on the manifold \mathcal{S} satisfying

$$\begin{aligned} 2\Omega D_i D^i \Omega - 3D_i \Omega D^i \Omega + \frac{1}{2}\Omega^2 r - 3\Sigma^2 - \frac{1}{2}\Omega^2 (K^2 - K_{ij}K^{ij}) + 2\Omega \Sigma K &= 0 \\ \Omega^3 D^i (\Omega^{-2} K_{ik}) - \Omega (D_k K - 2\Omega^{-1} D_k \Sigma) &= 0 \end{aligned}$$

where D is the covariant Levi-Civita connection of \mathbf{h} , r is its Ricci scalar and $K \equiv h^{ij}K_{ij}$. These equations are, respectively, the *conformal counterpart of the Hamiltonian and momentum constraints in vacuum*. In what follows, it will be assumed that \mathcal{S} is diffeomorphic to \mathbb{S}^3 and write $\mathcal{S} \approx \mathbb{S}^3$. As the initial hypersurface \mathcal{S} is compact, one can assume, without loss of generality, that $\Omega = 1$ and $\Sigma = 0$.

5.3 Wave maps

Since we are assuming $\mathcal{S} \approx \mathbb{S}^3$ there exists a diffeomorphism $\psi : \mathcal{S} \rightarrow \mathbb{S}^3$. The freedom encoded in the choice of diffeomorphism can be used to both fix the gauge and to discuss the parametrisation of the perturbation data in a appropriate setting.

The diffeomorphism ψ can be used to pull-back a coordinate system $x = (x^\alpha)$ in \mathbb{S}^3 to a coordinate system \hat{x} in \mathcal{S} since $\hat{x} = \psi^* x = x \circ \psi$. Exploiting the fact that ψ is a diffeomorphism we can define not only the pull-back $\psi^* : T^*\mathbb{S}^3 \rightarrow T^*\mathcal{S}$ but also the push-forward of its inverse $(\psi^{-1})_* : T\mathbb{S}^3 \rightarrow T\mathcal{S}$. Using this, we can push-forward vector fields \mathbf{c}_i on $T\mathbb{S}^3$ and pull-back their covector fields α^i on $T^*\mathcal{S}$ as

$$\hat{\mathbf{c}}_i = (\psi^{-1})_* \mathbf{c}_i, \quad \hat{\alpha}^i = \psi^* \alpha^i.$$

Now, let $\hat{\mathbf{e}}_i \equiv \hat{e}_i^j \hat{\mathbf{c}}_j$ be an \mathbf{h} -orthonormal frame and let us denote with \mathbf{D} and \mathcal{D} the Levi-Civita connection of \mathbf{h} and $\hat{\mathbf{h}}$ respectively. Observe that the covectors α^i and $\hat{\alpha}^i$ can be expanded in terms of their respective basis of 1-forms as

$$\alpha^i = \alpha^i_\alpha \mathbf{d}x^\alpha, \quad \hat{\alpha}^i = \hat{\alpha}^i_\alpha \mathbf{d}\hat{x}^\alpha.$$

Therefore, defining the spatial coordinate gauge source function as

$$f^i(x) \equiv D^j \hat{e}_j^i,$$

it can be shown that the right-hand side of the last equation can be computed to yield

$$f^i(x) = D^\beta \alpha^i_\beta = \alpha^i_\alpha D^\beta D_\beta x^\alpha + h^{\beta\gamma} \frac{\partial x^\alpha}{\partial \hat{x}^\beta} \frac{\partial x^\delta}{\partial \hat{x}^\gamma} \mathcal{D}_\delta \alpha^i_\alpha. \quad (35)$$

Finally, writing $f^\alpha(x) = f^i(x) \mathbf{c}_i^\alpha$ where $\mathbf{c}_i^\alpha = \langle \mathbf{d}x^\alpha, \mathbf{c}_i \rangle$ we find that

$$D^\beta D_\beta x^\alpha + \mathbf{c}_i^\alpha \left(h^{\beta\gamma} \frac{\partial x^\epsilon}{\partial \hat{x}^\beta} \frac{\partial x^\delta}{\partial \hat{x}^\gamma} \mathcal{D}_\delta \alpha^i_\epsilon - f^i(x) \right) = 0.$$

This last observation lead us to the notion of harmonic map. A diffeomorphism $\psi : \mathcal{S} \rightarrow \mathbb{S}^3$ is said to be a harmonic map if

$$h^{\beta\gamma} \frac{\partial x^\epsilon}{\partial \hat{x}^\beta} \frac{\partial x^\delta}{\partial \hat{x}^\gamma} \mathcal{D}_\delta \alpha^i_\epsilon = f^i(x). \quad (36)$$

Notice that the requirement (36) implies that $D^\beta D_\beta x^\alpha = 0$.

In the present analysis it will be assumed that ψ is the the identity map. Under this assumption the wave map condition (36) reduces to $h^{\beta\gamma} \mathcal{D}_\gamma \alpha^i_\beta = f^i(x)$. However, $\mathcal{D}_\gamma \alpha^i_\beta = -\gamma_\gamma^\delta \alpha^i_\delta$, which finally implies that

$$f^i(x) = -h^{\beta\gamma} \gamma_\gamma^\alpha \alpha^i_\beta.$$

Observe that for \mathbb{S}^3 we have that $\gamma_\beta^\alpha \gamma_\alpha^\gamma = -\epsilon_\beta^\gamma$, thus $f^i(x) = 0$.

Remark. The choice of the identity map ($\hat{x} = x$) for ψ means that we are going to use coordinates on \mathbb{S}^3 to coordinatise \mathcal{S} .

5.4 Data for a perturbation of the Milne Universe

The purpose of this section is to provide a description of initial data corresponding to perturbations of data for the Milne Universe. To this end, first notice from the last subsection that although the frame $\{\mathbf{c}_i\}$ is $\hat{\mathbf{h}}$ -orthonormal, it is not necessarily orthogonal respect to the intrinsic 3-metric \mathbf{h} on \mathcal{S} . Now, let $\{\mathbf{e}_i\}$ denote a \mathbf{h} -orthonormal frame over $T\mathcal{S}$ and let $\{\omega^i\}$ be the

associate cobasis. Now, assume that there exist vector fields $\{\check{e}_i\}$ such that an \mathbf{h} -orthonormal frame $\{e_i\}$ is related to an $\check{\mathbf{h}}$ -orthonormal frame \mathbf{c}_i through $e_i = \mathbf{c}_i + \check{e}_i$. This last requirement is equivalent to introducing coordinates on \mathcal{S} such that

$$\mathbf{h} = \mathring{\mathbf{h}} + \check{\mathbf{h}} = \check{\mathbf{h}} + \check{\check{\mathbf{h}}}. \quad (37)$$

Notice that the notation $\mathring{\cdot}$ is used to denote the value in the exact (background) solution while $\check{\cdot}$ is used to denote the perturbation.

In order to measure the size of the perturbed initial data, we introduce the Sobolev norms defined for any spinor quantity N_κ with κ being an arbitrary string of frame spinor indices, as

$$||N_\kappa||_{\mathcal{S}_*,m} \equiv \sum_{\kappa} ||N_\kappa||_{\mathbb{S}^3,m}$$

where \sum_{κ} is the sum over all the frame spinor indices encoded in κ and

$$||N_\kappa||_{\mathbb{S}^3,m} = \left(\sum_{l=0}^m \sum_{\alpha_1, \dots, \alpha_l}^3 \int_{\mathbb{S}^3} (\partial_{\alpha_1} \dots \partial_{\alpha_l} N_\kappa)^2 d^3x \right)^{1/2}.$$

Observe that since the indices in κ are frame indices, the quantities N_κ are scalars.

Suppose now that the initial spacelike hypersurface \mathcal{S} is described by the condition $\tau = \tau_0$. Then, restricting the results of Proposition 4 to the initial hypersurface \mathcal{S} , we get the following initial data for the wave equations of Proposition 1:

$$\begin{aligned} \mathring{\Xi}|_{\mathcal{S}} &= \cos \tau_0 + \cos \psi, & \mathring{e}_{AA'}b|_{\mathcal{S}} &= \sigma_{AA'}b, & \mathring{\Gamma}_{AA'BC}|_{\mathcal{S}} &= -\frac{i}{\sqrt{2}}\epsilon_{A(B}\tau_{C)A'}, \\ \mathring{\Phi}_{AA'BB'}|_{\mathcal{S}} &= \frac{1}{2}\tau_{AA'}\tau_{BB'} - \frac{1}{4}\epsilon_{AB}\epsilon_{AC}, & \Sigma_{AA'} &= -\sin \psi \sigma_{AA'}^3|_{\mathcal{S}}, & \mathring{\phi}_{ABCD}|_{\mathcal{S}} &= 0, \\ \mathring{s}|_{\mathcal{S}} &= -\frac{1}{2}(\cos \tau_0 + \cos \psi), & \mathcal{P}\mathring{\Xi}|_{\mathcal{S}} &= \Sigma|_{\mathcal{S}} = -\frac{1}{2}\sin \tau_0, & \mathcal{P}\mathring{e}_{AA'}b|_{\mathcal{S}} &= 0, \\ \mathcal{P}\mathring{\Gamma}_{AA'BC}|_{\mathcal{S}} &= 0, & \mathcal{P}\mathring{\Phi}_{AA'BB'}|_{\mathcal{S}} &= 0, \\ \mathcal{P}\mathring{\phi}_{ABCD}|_{\mathcal{S}} &= 0, & \mathcal{P}\mathring{s}|_{\mathcal{S}} &= \frac{1}{2}\sin \tau_0. \end{aligned}$$

In a manner consistent with the split (37), we make use of the above expressions to consider a perturbation of the initial data on \mathcal{S} of the form

$$\begin{aligned} \Xi|_{\mathcal{S}} &= \mathring{\Xi}|_{\mathcal{S}} + \check{\Xi}|_{\mathcal{S}}, & e_{AA'}b|_{\mathcal{S}} &= \mathring{e}_{AA'}b|_{\mathcal{S}} + \check{e}_{AA'}b|_{\mathcal{S}}, \\ \Gamma_{AA'BC}|_{\mathcal{S}} &= \mathring{\Gamma}_{AA'BC}|_{\mathcal{S}} + \check{\Gamma}_{AA'BC}|_{\mathcal{S}}, & \Phi_{AA'BB'}|_{\mathcal{S}} &= \mathring{\Phi}_{AA'BB'}|_{\mathcal{S}} + \check{\Phi}_{AA'BB'}|_{\mathcal{S}}, \\ \phi_{ABCD}|_{\mathcal{S}} &= \mathring{\phi}_{ABCD}|_{\mathcal{S}}, & s|_{\mathcal{S}} &= \mathring{s}|_{\mathcal{S}} + \check{s}|_{\mathcal{S}}, \\ \Sigma_{AA'}|_{\mathcal{S}} &= \mathring{\Sigma}_{AA'}|_{\mathcal{S}} + \check{\Sigma}_{AA'}|_{\mathcal{S}}. \end{aligned}$$

together with

$$\begin{aligned} \Sigma|_{\mathcal{S}} &= \mathring{\Sigma}|_{\mathcal{S}} + \check{\Sigma}|_{\mathcal{S}}, & \mathcal{P}e_{AA'}b|_{\mathcal{S}} &= \mathcal{P}\mathring{e}_{AA'}b|_{\mathcal{S}}, \\ \mathcal{P}\Gamma_{AA'BC}|_{\mathcal{S}} &= \mathcal{P}\mathring{\Gamma}_{AA'BC}|_{\mathcal{S}}, & \mathcal{P}\Phi_{AA'BB'}|_{\mathcal{S}} &= \mathcal{P}\mathring{\Phi}_{AA'BB'}|_{\mathcal{S}}, \\ \mathcal{P}\phi_{ABCD}|_{\mathcal{S}} &= \mathcal{P}\mathring{\phi}_{ABCD}|_{\mathcal{S}}, & \mathcal{P}s|_{\mathcal{S}} &= \mathcal{P}\mathring{s}|_{\mathcal{S}} + \mathcal{P}\check{s}|_{\mathcal{S}}. \end{aligned}$$

The above fields are solution to the equations implied by the initial data for the subsidiary system given in Proposition 2⁴. Observe that these expressions are consistent with equation (37).

⁴An example of how to explicitly obtain the equations satisfied by the data has to satisfy, was given in Section 4.2.1 for the case of the rescaled Weyl spinor. This construction required the use of a space spinor formalism. These equations are encoded in the requirement that the zero-quantities vanish on the initial hypersurface.

5.5 Perturbed solutions

Consistent with the discussion of the previous subsection, we will split the field unknowns into a background and a perturbation part. More precisely, we write

$$\begin{aligned}\Xi &= \mathring{\Xi} + \check{\Xi}, & \Sigma_{AA'} &= \mathring{\Sigma}_{AA'} + \check{\Sigma}_{AA'}, & e_{AA'}{}^b &= \mathring{e}_{AA'}{}^b + \check{e}_{AA'}{}^b, & s &= \mathring{s} + \check{s}, \\ \Gamma_{AA'}{}^B{}_C &= \mathring{\Gamma}_{AA'}{}^B{}_C + \check{\Gamma}_{AA'}{}^B{}_C, & \Phi_{AA'BB'} &= \mathring{\Phi}_{AA'BB'} + \check{\Phi}_{AA'BB'}, \\ \phi_{ABCD} &= \mathring{\phi}_{ABCD}.\end{aligned}$$

Following the notation used in the second remark in Section 3.8, we collect the independent components of the unknowns as a single vector-valued variable \mathbf{u} and write

$$\mathbf{u} = \mathring{\mathbf{u}} + \check{\mathbf{u}}.$$

The components of the contravariant metric tensor $g^{\mu\nu}(x, \mathbf{u})$ in the vector-valued wave equation (26) can be written as the metric for the background solution $\mathring{\mathbf{u}}$ plus a term depending on the unknown \mathbf{u}

$$g^{\mu\nu}(x; \mathbf{u}) = \mathring{g}^{\mu\nu}(x; \mathring{\mathbf{u}}) + \check{g}^{\mu\nu}(x; \mathbf{u}), \quad (38)$$

The latter can be expressed, alternatively, in spinorial terms as

$$g^{\mu\nu}(x; \mathbf{u}) = \epsilon^{AA'} \epsilon^{BB'} e_{AA'}{}^\mu e_{BB'}{}^\nu = \epsilon^{AA'} \epsilon^{BB'} (\mathring{e}_{AA'}{}^\mu \mathring{e}_{BB'}{}^\nu + \check{e}_{AA'}{}^\mu \check{e}_{BB'}{}^\nu). \quad (39)$$

Substituting the split (38) into equation (26) we get,

$$(\mathring{g}^{\mu\nu}(x; \mathring{\mathbf{u}}) + \check{g}^{\mu\nu}(x; \mathbf{u})) \partial_\mu \partial_\nu (\mathring{\mathbf{u}} + \check{\mathbf{u}}) + \mathbf{F}(x; \mathbf{u}, \partial \mathbf{u}) = 0.$$

Now, noticing that $\mathring{\mathbf{u}}$ is, in fact, a solution to

$$\mathring{g}^{\mu\nu}(x; \mathring{\mathbf{u}}) \partial_\mu \partial_\nu \mathring{\mathbf{u}} + \mathbf{F}(x; \mathring{\mathbf{u}}, \partial \mathring{\mathbf{u}}) = 0$$

it follows then that

$$\mathring{g}^{\mu\nu}(x; \mathring{\mathbf{u}}) \partial_\mu \partial_\nu \check{\mathbf{u}} + \check{g}^{\mu\nu}(x; \mathbf{u}) \partial_\mu \partial_\nu \mathring{\mathbf{u}} + \check{g}^{\mu\nu}(x; \mathbf{u}) \partial_\mu \partial_\nu \check{\mathbf{u}} + \mathbf{F}(x; \mathbf{u}, \partial \mathbf{u}) - \mathbf{F}(x; \mathring{\mathbf{u}}, \partial \mathring{\mathbf{u}}) = 0.$$

Finally, since the background solution $\mathring{\mathbf{u}}$ is known then the last equation can be recast as

$$(\mathring{g}^{\mu\nu}(x) + \check{g}^{\mu\nu}(x; \check{\mathbf{u}})) \partial_\mu \partial_\nu \check{\mathbf{u}} = \mathbf{F}(x; \check{\mathbf{u}}, \partial \check{\mathbf{u}}).$$

The above equation is in a form where the local existence and Cauchy stability theory of quasi-linear wave equations as given in, say, [15] can be applied. Notice that $\mathring{g}(x)$ is Lorentzian since it corresponds to the metric of the background solution —i.e. the metric of the Einstein Cosmos. Now, consider initial data $(\mathbf{u}_\star, \partial_t \mathbf{u}_\star)$ close enough to initial data $(\mathring{\mathbf{u}}_\star, \partial_t \mathring{\mathbf{u}}_\star)$ for the Milne Universe—that is, take

$$(\mathbf{u}_\star, \partial_t \mathbf{u}_\star) \in B_\varepsilon(\mathring{\mathbf{u}}_\star, \partial_t \mathring{\mathbf{u}}_\star), \quad (40)$$

where the notion of closeness is encoded in

$$B_\varepsilon(\mathbf{u}_\star, \mathbf{v}_\star) \equiv \{(\mathbf{w}_1, \mathbf{w}_2) \in H^m(\mathcal{S}, \mathbb{C}^N) \times H^m(\mathcal{S}, \mathbb{C}^N) \mid \|\mathbf{w}_1 - \mathbf{u}_\star\|_{\mathcal{S}, m} + \|\mathbf{w}_2 - \mathbf{v}_\star\|_{\mathcal{S}, m} \leq \varepsilon\}.$$

Using that $\mathbf{u} = \mathring{\mathbf{u}} + \check{\mathbf{u}}$, the requirement (40) is equivalent to saying that the initial data for the perturbation is small in the sense that

$$\|\check{\mathbf{u}}_\star\|_{\mathcal{S}, m} + \|\partial_t \check{\mathbf{u}}_\star\|_{\mathcal{S}, m} < \varepsilon. \quad (41)$$

With this remark in mind and recalling that $\mathring{\mathbf{u}}$ is explicitly known, observe that from equation (38) it follows that

$$g^{\mu\nu}(x; \mathbf{u}) = \mathring{g}^{\mu\nu}(x) + \check{g}^{\mu\nu}(x; \check{\mathbf{u}}).$$

Since the variable $\check{\mathbf{u}}_\star$ is a vector-valued function collecting the independent components of the conformal fields, and in particular $(\check{\epsilon}_{\mathbf{A}\mathbf{A}'})_\star$, it follows that for sufficiently small initial $\check{\mathbf{u}}_\star$ the perturbation $\check{g}^{\mu\nu}$ will be small. Therefore, choosing ε small enough we can guarantee that the metric $\check{g}^{\mu\nu}(x) + \check{g}^{\mu\nu}(x; \check{\mathbf{u}}_\star)$ is initially Lorentzian. To pose an initial value problem for these equations we have to fix the initial data sets that are to be considered. In this case, we are interested in hyperboloidal initial data sets. The reason behind this choice will be clarified in the sequel. In what follows, consider initial data \mathbf{u}_\star for the conformal Einstein equations prescribed on an hypersurface

$$\mathcal{H} \equiv \{p \in \mathbb{R} \times \mathbb{S}^3 \mid 0 \leq \psi(p) \leq \pi - \tau_0, \tau(p) = \tau_0\}$$

with boundary

$$\mathcal{Z} \equiv \{p \in \mathbb{R} \times \mathbb{S}^3 \mid \psi(p) = \pi - \tau_0, \tau(p) = \tau_0\}$$

for some constant $\tau_0 > 0$ and where, recalling the discussion of Section 5.3, the coordinates (x^α) on \mathbb{S}^3 are being used to coordinatise \mathcal{S} . Therefore \mathcal{H} is to be regarded as a subset of $\mathcal{S} \approx \mathbb{S}^3$. The initial data is said to be an hyperboloidal data set if

$$\Xi|_{\mathcal{H} \setminus \mathcal{Z}} > 0, \quad \Xi|_{\mathcal{Z}} = 0, \quad (\Sigma_a \Sigma^a)|_{\mathcal{Z}} = 0, \quad \mathcal{P}\Xi|_{\mathcal{Z}} = \Sigma < 0. \quad (42)$$

The above conditions tell us, *a priori*, that Ξ acts as a boundary defining function which vanishes on \mathcal{Z} . This is a key observation that will be used in Section 5.6 where the conformal structure of the solutions is to be analysed. To arrive to a similar conclusion about Σ let us shift the coordinate τ by an amount of $\frac{1}{2}\pi$, namely $\hat{\tau} = \tau - \frac{1}{2}\pi$.

In the case of the background solution $\hat{\mathbf{u}}$ we can analogously define an hypersurface $\hat{\mathcal{H}}$ with boundary $\hat{\mathcal{Z}}$ where $\hat{\mathcal{H}}$ is as a region of \mathbb{S}^3 satisfying analogous conditions as those given in (42). In that case, the initial hypersurface $\hat{\tau} = 0$ corresponds to the so-called *standard Minkowski hyperboloid*. The conformal factor for the background solution, —i.e. the Milne spacetime is given by

$$\hat{\Xi} = \cos \psi - \sin \hat{\tau}.$$

Therefore

$$\hat{\Sigma} = \mathcal{P}\hat{\Xi} = -\cos \hat{\tau}, \quad \hat{\Sigma}_3 = -\sin \psi.$$

Observe that $\hat{\Sigma} < 0$ for $\hat{\tau} \in (0, \frac{1}{2}\pi)$ and $\hat{\Sigma} > 0$ for $\hat{\tau} \in (\frac{1}{2}\pi, \frac{3}{2}\pi)$. Thus, recalling that $\Sigma = \hat{\Sigma} + \check{\Sigma}$ then $\Sigma < 0$ for $\hat{\tau} \in (0, \frac{1}{2}\pi)$ and $\Sigma > 0$ for $\hat{\tau} \in (\frac{1}{2}\pi, \frac{3}{2}\pi)$ for ε small enough, therefore there is at least one point where $\Sigma = 0$.

Now, recall that the initial data $(\mathbf{u}_\star, \partial_\tau \mathbf{u}_\star)$ is only defined in the initial hypersurface \mathcal{H} while the data $(\check{\mathbf{u}}_\star, \partial_\tau \check{\mathbf{u}}_\star)$ is defined in the whole of $\mathcal{S} \approx \mathbb{S}^3$. To be able to apply the theory of [15] we need to extend the data $(\mathbf{u}_\star, \partial_\tau \mathbf{u}_\star)$ to \mathcal{S}_\star . In what follows let $\mathbf{w}_\star = (\mathbf{u}_\star, \partial_\tau \mathbf{u}_\star)$. We can extend the initial data by invoking the *Extension Theorem* which states that there exists a linear operator

$$\mathcal{E} : H^m(\mathcal{H}, \mathbb{C}^N) \rightarrow H^m(\mathbb{S}^3, \mathbb{C}^N)$$

such that if $\mathbf{w}_\star \in H^m(\mathcal{H}, \mathbb{C}^N)$ then $\mathcal{E}\mathbf{w}_\star(x) = \mathbf{w}_\star(x)$ almost everywhere in \mathcal{H} and

$$\|\mathcal{E}\mathbf{w}_\star\|_{m, \mathcal{S}} \leq K \|\mathbf{w}_\star\|_{m, \mathcal{H}}$$

where K is a universal constant for fixed m —see e.g. [26]. Hence, using equation (41) we can make $\|\mathcal{E}\mathbf{w}_\star\|_{m, \mathcal{S}}$ small as necessary by making ε small —that is, the size of the extended data is controlled by the data in the initial hypersurface \mathcal{H} . Therefore, the extended data will be given by

$$\mathcal{E}\mathbf{u}_\star = \hat{\mathbf{u}}_\star + \mathcal{E}\check{\mathbf{u}}_\star \quad \mathcal{E}\partial_\tau \mathbf{u}_\star = \partial_\tau \hat{\mathbf{u}}_\star + \mathcal{E}\partial_\tau \check{\mathbf{u}}_\star$$

which are well defined on $H^m(\mathbb{S}^3, \mathbb{C}^N)$. Using equation (41) we observe that

$$\|\mathcal{E}\check{\mathbf{u}}_\star\|_{\mathcal{H}, m} + \|\mathcal{E}\partial_\tau \check{\mathbf{u}}_\star\|_{\mathcal{H}, m} \leq K\varepsilon.$$

Remark. The fact that the extension of the data obtained in the previous paragraph is not unique and it does not necessarily satisfy the constraints of Proposition 3 is not a problem in our analysis since

$$D^+(\mathcal{H}) \cap I^+(\mathcal{S} \setminus \mathcal{H}) = \emptyset.$$

The proof of the last statement follows by contradiction. Let $q \in D^+(\mathcal{H}) \cap I^+(\mathcal{S} \setminus \mathcal{H})$. Then, in the one hand we have that $q \in I^+(\mathcal{S} \setminus \mathcal{H})$, so that it follows that there exists a future timelike curve γ from $p \in \mathcal{S} \setminus \mathcal{H}$ to q . On the other hand $q \in D^+(\mathcal{H})$ which means that every past in extendible causal curve through q intersects \mathcal{H} , therefore $p \in \mathcal{H}$. This is a contradiction since $p \in \mathcal{S} \setminus \mathcal{H}$.

We are now in position to make use of a local existence and Cauchy stability result adapted from [15] —see Appendix E, to establish the following theorem:

Theorem 1 (*Existence and Cauchy stability*). *Let $(\mathbf{u}_\star, \partial_\tau \mathbf{u}_\star) = (\mathring{\mathbf{u}}_\star + \check{\mathbf{u}}_\star, \partial_\tau \mathring{\mathbf{u}}_\star + \partial_\tau \check{\mathbf{u}}_\star)$ be hyperboloidal initial data for the conformal wave equations on an 3-dimensional manifold \mathcal{H} where $(\mathring{\mathbf{u}}_\star, \partial_\tau \mathring{\mathbf{u}}_\star)$ denotes initial data for the Milne Universe. Let $(\mathcal{E}\mathbf{u}_\star, \mathcal{E}\partial_\tau \mathbf{u}_\star)$ denote the extension of these data to $\mathcal{S} \approx \mathbb{S}^3$. Then, for $m \geq 4$ and $\hat{\tau}_\bullet \geq \frac{3}{2}\pi$ there exist an $\varepsilon > 0$ such that:*

(i) *For $\|\check{\mathbf{u}}_\star\|_{\mathcal{H},m} + \|\partial_\tau \check{\mathbf{u}}_\star\|_{\mathcal{H},m} < \varepsilon$, there exist a unique solution $\mathbf{u} = \mathring{\mathbf{u}} + \check{\mathbf{u}}$ to the wave equations of Proposition 1 with a minimal existence interval $[0, \hat{\tau}_\bullet]$ and $\mathbf{u} \in C^{m-2}([0, \hat{\tau}_\bullet \times \mathcal{S}], \mathbb{C}^N)$.*

(ii) *Given a sequence $(\mathbf{u}_\star^{(n)}, \mathbf{v}_\star^{(n)}) \in B_\varepsilon(\mathbf{u}_\star, \mathbf{v}_\star) \cap D_\delta$ such that*

$$\|\mathbf{u}_\star^{(n)} - \mathbf{u}_\star\|_{\mathcal{S},m} \rightarrow 0, \quad \|\mathbf{v}_\star^{(n)} - \mathbf{v}_\star\|_{\mathcal{S},m} \rightarrow 0, \quad \text{as } n \rightarrow \infty,$$

then for the solutions $\mathbf{u}^{(n)}$ with $\mathbf{u}^{(n)} = \mathbf{u}_\star^{(n)}$ and $\partial_t \mathbf{u}^{(n)}(0, \cdot) = \mathbf{v}_\star^{(n)}$, it holds that

$$\|\mathbf{u}^{(n)}(t, \cdot) - \mathbf{u}(t, \cdot)\|_{\mathcal{S},m} \rightarrow 0 \quad \text{as } n \rightarrow \infty$$

uniformly in $t \in [0, \tau_\bullet)$ as $n \rightarrow \infty$.

(iii) *The solution $\mathbf{u} = \mathring{\mathbf{u}} + \check{\mathbf{u}}$ is unique in $D^+(\mathcal{H})$ and implies, wherever $\Xi \neq 0$, a C^{m-2} solution to the Einstein vacuum equations with vanishing cosmological constant.*

Proof. Points (i) and (ii) are a direct application of Theorem 2 given in Appendix E. The condition ensuring that $\dot{\mathbf{g}}(x) + \dot{\mathbf{g}}(x; \mathbf{u})$ is Lorentzian is encoded in the requirement of the perturbation for the initial data being small as discussed in Section 5.5.

The statement of point (iii) follows from the discussion of Section 4 for the propagation of the constraints and the subsidiary system as summarised in Propositions 1 and 3. In particular, in this section it was shown that a solution to the spinorial wave equations is a solution to the conformal Einstein field equations if initial data satisfies the appropriate conditions. As exemplified in Section 4.2.1 for the rescaled Weyl spinor, requiring the zero-quantities to vanish in the initial hypersurface renders conditions on the initial data. Finally, recall that a solution to the CEFE implies a solution to the Einstein field equations wherever $\Xi \neq 0$ —see [8]. \square

5.6 Structure of the conformal Boundary

Now, we will complement Theorem 1 by showing that the conformal boundary \mathcal{S} coincides with the Cauchy horizon of \mathcal{H} . The argument of this section is based on analogous discussion in [10]. Since the Cauchy horizon $H(\mathcal{H}) = \partial(D^+(\mathcal{H}))$ is generated by null geodesics with endpoints on \mathcal{Z} the null generators of $H(\mathcal{H})$ —i.e the null vectors tangent to $H(\mathcal{H})$ — are given at \mathcal{Z} by $\Sigma_{\mathbf{a}}|_{\mathcal{Z}}$ as it follows by the initial hyperboloidal data (42). We then define two null vectors (\mathbf{n}, \mathbf{l}) on \mathcal{Z} by setting

$$l_{\mathbf{a}\star} = \Sigma_{\mathbf{a}}|_{\mathcal{Z}}, \quad \mathbf{n}_\star \perp \mathcal{Z}, \quad g(\mathbf{n}_\star, \mathbf{l}_\star) = 1 \quad \text{on } \mathcal{Z}. \quad (43)$$

We complement this pair of null vectors $\{\mathbf{l}_*, \mathbf{n}_*\}$, where \mathbf{l}_* is tangent to $H(\mathcal{H})$ on \mathcal{Z} and \mathbf{n}_* is normal to \mathcal{Z} , with a pair of complex conjugate vectors \mathbf{m}_* and $\bar{\mathbf{m}}_*$ tangent to \mathcal{Z} such that $g(\mathbf{m}_*, \bar{\mathbf{m}}_*) = 1$, so as to obtain the tetrad $\{\mathbf{l}_*, \mathbf{n}_*, \mathbf{m}_*, \bar{\mathbf{m}}_*\}$. In order to obtain a Newman-Penrose frame $\{\mathbf{l}, \mathbf{n}, \mathbf{m}, \bar{\mathbf{m}}\}$ off \mathcal{Z} along the null generators of $H(\mathcal{H})$ we propagate them by parallel transport in the direction of \mathbf{l} by requiring

$$l^a \nabla_a l^b = 0, \quad l^a \nabla_a n^b = 0, \quad l^a \nabla_a m^b = 0. \quad (44)$$

Now, suppose that we already have a solution to the conformal wave equations. Using the result of Proposition 3, we know that the solution will also satisfy the CEFÉ. In this section we will make use of the CEFÉ equations to study the conformal boundary. From the tensorial (frame) version of the CEFÉ as given in Appendix D, one notices the following subset consisting of equations (67d), (67e) and the definition of Σ_a as the gradient of the conformal factor:

$$\nabla_a \Xi = \Sigma_a, \quad (45a)$$

$$\nabla_a \Sigma_b = s g_{ab} - \Xi L_{ab}, \quad (45b)$$

$$\nabla_a s = -L_{ab} \Sigma^b. \quad (45c)$$

Transvecting the first two equations, respectively, with l^a , $l^a l^b$ and $l^a m^b$ we get

$$\begin{aligned} l^a \nabla_a \Xi &= l^a \Sigma_a, \\ l^a \nabla_a (l^b \Sigma_b) &= -\Xi L_{ab} l^a l^b, \\ l^a \nabla_a (m^b \Sigma_b) &= -\Xi L_{ab} l^a m^b, \end{aligned}$$

where we have used (44) and the fact that \mathbf{l} is null and orthogonal to \mathbf{m} . The latter equations can be read as a system of homogeneous transport equations along the integral curves of \mathbf{l} for a vector-valued variable containing as components Ξ , $\Sigma_a l^a$ and $\Sigma_a m^a$. Written in matricial form one has

$$\nabla_l \begin{pmatrix} \Xi \\ \Sigma_a l^a \\ \Sigma_a m^a \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ -L_{cd} l^c l^d & 0 & 0 \\ -L_{cd} l^c m^d & 0 & 0 \end{pmatrix} \begin{pmatrix} \Xi \\ \Sigma_a l^a \\ \Sigma_a m^a \end{pmatrix} \quad (46)$$

Observe that the column vector shown in the last equation is zero on \mathcal{Z} , since $\Xi|_{\mathcal{Z}} = 0$, $(l^a \Sigma_a)|_{\mathcal{Z}} = (l^a l_a)|_{\mathcal{Z}} = 0$ and $(\Sigma_a m^a)|_{\mathcal{Z}} = (l_a m^a)|_{\mathcal{Z}} = 0$ which follows from (42) and (43). Since equation (46) is homogeneous and it has vanishing initial data on \mathcal{Z} we have that Ξ , $\Sigma_a l^a$ and $\Sigma_a m^a$ will be zero along \mathbf{l} until one reaches a caustic point. Consequently, we conclude that the conformal Ξ factor vanishes in the portion of $H(\mathcal{H})$ which is free of caustics. Thus, this portion of $H(\mathcal{H})$ can be interpreted as the conformal boundary of the physical spacetime $(\tilde{\mathcal{M}}, \tilde{g})$. In addition, notice that from the vanishing of the column vector of equation (46) it follows that $\Sigma_a l^a = \Sigma_a m^a = 0$ on $H(\mathcal{H})$. Therefore, the only component of Σ_a that can be different from zero is $\Sigma_a n^a$. Accordingly, Σ^a is parallel to l^a and $\Sigma^a = (\Sigma_c n^c) l^a$. Moreover, since $g(\mathbf{n}_*, \mathbf{l}_*) = 1$ it follows that $(\Sigma_a n^a)|_{\mathcal{Z}} = 1$ ⁵.

Now, in order to extract the information contained in $\Sigma_a n^a$ one transvects (45b) with $l^a n_b$, to obtain

$$l^a \nabla_a (n^b \Sigma_b) = s g_{ab} l^a n^b - \Xi L_{ab} l^a n^b.$$

Using that $g(\mathbf{l}, \mathbf{n}) = 1$ and that Ξ vanishes on $H(\mathcal{H})$ one concludes that

$$\nabla_l (\Sigma_a n^a) = s \quad \text{on } H(\mathcal{H}).$$

We can obtain a further equation transvecting (45c) with l^a

$$l^a \nabla_a s = -L_{ab} l^a \Sigma^b = -L_{ab} l^a \Sigma_f n^f l^b \quad \text{on } H(\mathcal{H}).$$

⁵This can also be shown by noticing that $(n^b \Sigma_b)|_{\mathcal{Z}} = (n^b l_b)|_{\mathcal{Z}} = 1$.

It follows then that one has the system

$$\nabla_l \begin{pmatrix} \Sigma_a n^a \\ s \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -L_{cd} l^c l^d & 0 \end{pmatrix} \begin{pmatrix} \Sigma_a n^a \\ s \end{pmatrix}.$$

Since $(\Sigma_a n^a)|_{\mathcal{Z}} = 1$ (i.e. non-vanishing), the solution for the column vector formed by s and $\Sigma_a n^a$ cannot be zero. Accordingly, s and $\Sigma_a n^a$ cannot vanish simultaneously. Finally, transvecting equation (45b) with $m^a \bar{m}^b$ we get

$$m^a \bar{m}^b \nabla_a \Sigma_b = -\Xi m^a \bar{m}^b L_{ab} + s g_{ab} m^a \bar{m}^b.$$

Using that $g(\mathbf{m}, \bar{\mathbf{m}}) = 1$ and restricting to $H(\mathcal{H})$ where $\Xi = 0$ we obtain

$$\bar{m}^b m^a \nabla_a \Sigma_b = s \quad \text{on } H(\mathcal{H}).$$

Using $g(\bar{\mathbf{m}}, \mathbf{l}) = 0$ it follows that the left hand side of the last equation is equivalent to

$$\begin{aligned} m^a \bar{m}^b \nabla_a \Sigma_b &= m^a \bar{m}^b \nabla_a (\Sigma_c n^c l_b) \\ &= m^a \bar{m}^b \Sigma_c n^c \nabla_a l_b + m^a \bar{m}^b l_b \nabla_a \Sigma_c n^c \\ &= \Sigma_c n^c m^a \bar{m}^b \nabla_a l_b. \end{aligned}$$

Finally, recalling the definition of the expansion $\rho \equiv -m^a \bar{m}^b \nabla_a l_b$ (in the Newman-Penrose notation [24]) we finally obtain

$$\Sigma_a n^a \rho = -s \quad \text{on } H(\mathcal{H}).$$

We already know that the only possible non-zero component of the gradient of Ξ is $\Sigma_a n^a$ and that it cannot vanish simultaneously with s . This means that $\mathbf{d}\Xi = 0$ implies $\rho \rightarrow \infty$ on $H(\mathcal{H})$. To be able to identify the point i^+ where $\mathbf{d}\Xi = 0$ with timelike infinity we need to calculate the Hessian of the conformal factor. Observe that this information is contained in the conformal field equation (67d). Considering this equation at $H(\mathcal{H})$, where we have already shown that the conformal factor vanishes, we get

$$\nabla_a \nabla_b \Xi = s g_{ab}.$$

Now, as we have shown that s and $\Sigma_a n^a$ (or, equivalently, $\mathbf{d}\Xi$) do not vanish simultaneously we conclude that $s \neq 0$ and that $\nabla_a \nabla_b \Xi$ is non-degenerate. Thus, we can consider the point i^+ on $H(\mathcal{H})$ where both Ξ and $\mathbf{d}\Xi$ vanish as representing future timelike infinity for the physical spacetime $(\tilde{\mathcal{M}}, \tilde{g})$.

Remark. Observe that the construction discussed in the previous paragraphs crucially assumes that Ξ_* is zero on the boundary \mathcal{Z} of the initial hypersurface \mathcal{H} . This construction cannot be repeated if we were to take another hypersurface \mathcal{H}' with boundary \mathcal{Z}' where the conformal factor does not vanish. This is the case of an initial hypersurface that intersects the cosmological horizon, where for the reference solution the conformal factor does not vanish —see Figure 2.

The results of the analysis of this section are summarised in the following:

Proposition 5. (Structure of the conformal boundary) *Let \mathbf{u} denote a solution to the conformal wave equations constructed as described in Theorem 1, then, there exists a point $i^+ \in H(\mathcal{H})$ where $\Xi|_{i^+} = 0$ and $\mathbf{d}\Xi|_{i^+} = 0$ but the Hessian $\nabla_a \nabla_b \Xi|_{i^+}$ is non-degenerate. In addition, $\mathbf{d}\Xi \neq 0$ on $\mathcal{I}^+ = H(\mathcal{H}) \setminus \{i^+\}$. Moreover $D^+(\mathcal{H}) = J^-(i^+)$.*

Proof. From the conclusions of Theorem 1 and the discussion of Section 5.6 it follows that if we have a solution to the conformal wave equations which, in turn implies a solution to the conformal field equations, then there exists a point i^+ in $H(\mathcal{H})$ where both the conformal factor and its gradient vanish but $\nabla_a \nabla_b \Xi$ is non-degenerate. This means that i^+ can be regarded as future timelike infinity for the physical spacetime. In addition, null infinity \mathcal{I}^+ will be located at $H(\mathcal{H}) \setminus \{i^+\}$ where the conformal factor vanishes but its gradient does not. \square

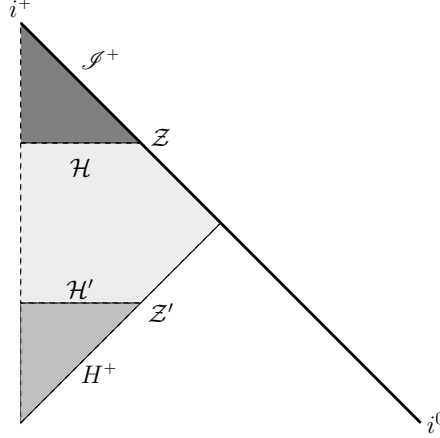


Figure 2: Portion of the Penrose diagram of the Milne Universe showing the initial hypersurface \mathcal{H} where the hyperboloidal data is prescribed. At \mathcal{Z} the conformal factor vanishes and the argument of Section 5.6 can be applied. The dark gray area represents the development of the data on \mathcal{H} . Compare with the case of the hypersurface \mathcal{H}' which intersects the horizon at \mathcal{Z}' where the argument cannot be applied. Analogous hypersurfaces can be depicted for the lower diamond of the complete diagram of Figure 1.

6 Conclusions

In this article we have shown that the spinorial frame version of the CEFÉ implies a system of quasilinear wave equations for the various conformal fields. The use of spinors allows a systematic and clear deduction of the equations and the not less important issue of the propagation of the constraints. The fact that the metric is not part of the unknowns in the spinorial formulation of the CEFÉ simplifies the considerations of hyperbolicity of the operator $\square \equiv \nabla_a \nabla^a$. The application of these equations to study the semiglobal stability of the Milne Universe exemplifies how the extraction of a system of quasilinear wave equations out of the CEFÉ allows to readily make use of the general theory of partial differential equations to obtain non-trivial statements about the global existence of solutions to the Einstein field equations. The analysis of the present article has been restricted to the vacuum case. However, a similar procedure can be carried out, in the non-vacuum case, for some suitable matter models with trace-free energy-momentum tensor —see e.g. [11].

In addition, the present analysis has been restricted to the case of the so-called standard CEFÉ. There exists another more general version of the CEFÉ, the so-called, extended conformal Einstein field equations (XCEFÉ) in which the various equations are expressed in terms of a Weyl connection —i.e. a torsion free connection which is not necessarily metric but, however, respects the structure of the conformal class, [12]. The hyperbolic reduction procedures for the XCEFÉ available in the literature do not make use of gauge source functions. Instead, one makes use of conformal Gaussian systems based on the congruence of privileged curves known as conformal geodesics to extract a first order symmetric hyperbolic system. It is an interesting open question to see whether it is possible to use conformal Gaussian systems to deduce wave equations for the conformal fields in the XCEFÉ.

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A Appendix: spinorial relations

In this appendix we recall several relations and identities that are used repeatedly throughout this article —see [21]. In addition, using the remarks made in [20] we give a generalisation for the spinorial Ricci identities for a connection which is metric but not necessarily torsion free.

A.1 The Levi-Civita case

In this subsection we recall some well known relations satisfied the curvature spinors of a Levi-Civita connection. The discussion of this subsection follows [21]. First recall the decomposition of a general curvature spinor

$$\dot{R}_{AA'BB'CC'DD'} = \dot{R}_{ABCC'DD'}\epsilon_{B'A'} + \dot{R}_{A'B'CC'DD'}\epsilon_{BA}.$$

We can further decompose the reduced spinor $\dot{R}_{ABCC'DD'}$ as

$$\dot{R}_{ABCC'DD'} = \dot{X}_{ABCD}\epsilon_{C'D'} + \dot{Y}_{ABC'D'}\epsilon_{CD},$$

where

$$\dot{X}_{ABCD} \equiv \frac{1}{2}\dot{R}_{AB(C|E'|D)}^{E'} \quad \dot{Y}_{ABCD} \equiv \frac{1}{2}\dot{R}_{ABE(C'|D')}^{E'}.$$

In the above expressions the symbol $\dot{}$ over the Kernel letter indicates that this relation is general —i.e. the connection is not necessarily neither metric nor torsion-free. The spinors \dot{X}_{ABCD} and \dot{Y}_{ABCD} are not necessarily symmetric in AB .

It is well known that if that the connection is metric, then the spinors \widehat{X}_{ABCD} and $\widehat{Y}_{ABC'D'}$ have the further symmetries

$$\widehat{X}_{ABCD} = \widehat{X}_{(AB)CD}, \quad \widehat{Y}_{ABC'D'} = \widehat{Y}_{(AB)C'D'}. \quad (47)$$

We add the symbol $\widehat{}$ over the Kernel letter to denote that only the metricity of the connection is being assumed. Now, if the connection is not only metric but, in addition, is torsion free (i.e. it is a Levi-Civita connection) then the *first Bianchi identity* $R_{a[bcd]} = 0$ can be written in spinorial terms as

$$R_{AB}{}^{CD}{}_{A'B'}{}^{D'C'} = 0,$$

which implies that $X_{SP}{}^{SP} = \bar{X}_{S'P'}{}^{S'P'}$. Accordingly $X_{SP}{}^{SP}$ is a real scalar and $Y_{ABA'B'}$ is a Hermitian spinor which, following the notation of [21], will be denoted by $\Phi_{ABA'D'} = \bar{\Phi}_{A'B'AB}$. Collecting all this information and decomposing in terms of irreducible components one obtains the usual decomposition of the curvature spinors

$$X_{ABCD} = \Psi_{ABCD} + \Lambda(\epsilon_{DB}\epsilon_{CA} + \epsilon_{CB}\epsilon_{DA}), \quad Y_{ABC'D'} = \Phi_{ABC'D'},$$

where Ψ_{ABCD} is the Weyl spinor and $\Phi_{ABC'D'}$ is the Ricci spinor. The latter is the spinorial counterpart of a world tensor (because of its Hermiticity which is consequence of the first Bianchi identity) and Λ is a real scalar (consequence of the first Bianchi identity again). Additionally, observe that one also has that $X_{A(BC)}{}^A = 0$. This is a consequence of the interchange of pairs symmetry $R_{abcd} = R_{cdab}$ of the Riemann tensor of a Levi-Civita connection. For a general connection the right hand side of last equation is not necessarily zero⁶.

In the Levi-Civita case, the spinorial Ricci identities are the spinorial counterpart of

$$[\nabla_a \nabla_b]v^c = R^c{}_{eab}v^e.$$

⁶ The reason is that the interchange of pairs symmetry is a consequence of the antisymmetry in the first and second pairs of indices and the first Bianchi identity which for a general connection will involve the torsion and its derivatives.

These identities are given in terms of the operator $\square_{AB} = \nabla_{Q(A}\nabla_{B)}^Q$. The spinorial Ricci identities are given in a rather compact form by

$$\square_{AB}\xi^C = X_{ABQ}{}^C\xi^Q, \quad \square_{A'B'}\xi^C = \Phi_{A'B'Q}{}^C\xi^Q, \quad (48)$$

$$\square_{AB}\xi_C = -X_{ABC}{}^Q\xi_Q, \quad \square_{A'B'}\xi_C = -\Phi_{A'B'C}{}^Q\xi_Q. \quad (49)$$

It is useful to combine these identities with the decomposition of X_{ABCD} to obtain a more detailed list of relations. The following expressions are repeatedly used in this article:

$$\square_{AB}\xi_C = \Psi_{ABCQ}\xi^Q - 2\Lambda\xi_{(A}\epsilon_{B)C}, \quad \square_{(AB}\xi_{C)} = \Psi_{ABCQ}\xi^Q, \quad (50)$$

$$\square_{AB}\xi^B = -3\Lambda\xi_A, \quad \square_{A'B'}\xi_C = \xi^Q\Phi_{QCA'B'}. \quad (51)$$

Using these relations and the Jacobi identity (ϵ -identity) the second Bianchi identity can be expressed in terms of spinors as

$$\nabla^A{}_{B'}X_{ABCD} = \nabla^{A'}{}_B\Phi_{CDA'B'}.$$

For completeness, we recall the relation between $\Phi_{ABA'B'}$ and the Ricci tensor and between R and Λ . Namely, that

$$R_{ac} \mapsto R_{AA'CC'} = 2\Phi_{ACA'C'} - 6\Lambda\epsilon_{AC}\epsilon_{A'C'}, \quad R = -24\Lambda. \quad (52)$$

From the above expressions it follows that $2\Phi_{ABA'B'}$ is the spinorial counterpart of the trace-free Ricci tensor $R_{\{ab\}} \equiv R_{ab} - \frac{1}{4}Rg_{ab}$. From this last observation, it follows that the spinorial counterpart of the Schouten tensor can be rewritten in terms of $\Phi_{AA'BB'}$ and Λ . Recalling the definition of the 4-dimensional Schouten tensor $L_{ab} = \frac{1}{2}R_{ab} - \frac{1}{12}Rg_{ab}$ and equation (52) we get

$$L_{ABA'B'} = \Phi_{ACA'C'} - \Lambda\epsilon_{AC}\epsilon_{A'C'}. \quad (53)$$

The second contracted Bianchi identity can be recast in terms of these spinors as

$$\nabla^{CA'}\Phi_{CDA'B'} + 3\nabla_{DB'}\Lambda = 0. \quad (54)$$

A.2 Spinorial Ricci identities for a metric connection

We now consider the case of a connection $\widehat{\nabla}$ which is metric but not torsion free. First, we need to obtain a suitable generalisation of the operator \square_{AB} . In order to achieve this, observe that the relation $[\nabla_a, \nabla_b]u^d = R^d{}_{cab}u^c$ valid for a Levi-Civita connection extends to a connection with torsion as

$$[\widehat{\nabla}_a, \widehat{\nabla}_b]u^d = \widehat{R}^d{}_{cab}u^c + \Sigma_a{}^c{}_b\widehat{\nabla}_cu^d.$$

Another way to think the last equation is to define a modified commutator of covariant derivatives through

$$[\widehat{\nabla}_a, \widehat{\nabla}_b]u^d \equiv ([\widehat{\nabla}_a, \widehat{\nabla}_b] - \Sigma_a{}^c{}_b\widehat{\nabla}_c)u^d.$$

In this way we can recast the Ricci identities as

$$[\widehat{\nabla}_a, \widehat{\nabla}_b]u^d = \widehat{R}^d{}_{cab}u^c.$$

This observation leads us to an expression for the generalised operator

$$\widehat{\square}_{AB} \equiv \widehat{\nabla}_{C'(A}\widehat{\nabla}_{B)}^{C'}.$$

The relation between this operator and the commutator of covariant derivatives is

$$[\widehat{\nabla}_{AA'}, \widehat{\nabla}_{BB'}] = \epsilon_{A'B'}\widehat{\square}_{AB} + \epsilon_{AB}\widehat{\square}_{A'B'}.$$

We cannot directly write down the equivalent spinorial Ricci identities simply by replacing X and Y by \widehat{X} and \widehat{Y} because of appearance of the term $\Sigma_a{}^c{}_b \widehat{\nabla}_c u^d$ in the definition of the curvature tensor. A way to get around this difficulty is to define a modified operator $\widehat{\Pi}_{AB}$ formed using the modified commutator of covariant derivatives instead of the usual commutator. In this way we can directly translate the previous formulae simply by replacing X and Y by \widehat{X} and \widehat{Y} . Now, the relation between $\widehat{\Pi}_{AB}$ and $\widehat{\Pi}_{AB}$ can be obtained by observing that

$$\begin{aligned}\widehat{\Pi}_{CD} &= \frac{1}{2} \epsilon^{C'D'} [\widehat{\nabla}_{CC'}, \widehat{\nabla}_{DD'}] \\ &= \frac{1}{2} \epsilon^{C'D'} ([\widehat{\nabla}_{CC'}, \widehat{\nabla}_{DD'}] - \Sigma_{CC'}{}^{EE'}{}_{DD'} \widehat{\nabla}_{EE'}) \\ &= \frac{1}{2} (\widehat{\nabla}_{D'C} \widehat{\nabla}_D{}^{D'} + \widehat{\nabla}_{D'D} \widehat{\nabla}_C{}^{D'} - \Sigma_{CD'}{}^{EE'}{}_D{}^{D'} \widehat{\nabla}_{EE'}).\end{aligned}\quad (55)$$

Using the antisymmetry of the torsion spinor we have the decomposition

$$\Sigma_{AA'}{}^{CC'}{}_{BB'} = \epsilon_{AB} \Sigma_A{}^{EE'}{}_B + \epsilon_{A'B'} \Sigma_{A'}{}^{EE'}{}_{B'}, \quad (56)$$

where the reduced spinor is given by $\Sigma_A{}^{EE'}{}_B = \frac{1}{2} \Sigma_{(A|Q'|}{}^{EE'}{}_{B)} Q'^{Q'}$. Using this decomposition and symmetrising expression (55) in the indices CD in one obtains

$$\widehat{\Pi}_{CD} = \widehat{\nabla}_{D'(C} \widehat{\nabla}_{D)}{}^{D'} - \Sigma_C{}^{EE'}{}_D \widehat{\nabla}_{EE'} = \widehat{\Pi}_{CD} - \Sigma_C{}^{EE'}{}_D \widehat{\nabla}_{EE'}.$$

Therefore

$$\widehat{\Pi}_{AB} = \widehat{\Pi}_{AB} + \Sigma_A{}^{EE'}{}_B \widehat{\nabla}_{EE'}. \quad (57)$$

In order to compute explicitly how $\widehat{\Pi}_{AB}$ acts on spinors we only need to compute the generalised the spinors \widehat{X}_{ABCD} and $\widehat{\Phi}_{ABC'D'}$.

As discussed in previous paragraphs, the fact that the connection is not torsion free is reflected in the symmetries of the curvature spinors. We still have that the symmetries in (47) hold due to the metricity of $\widehat{\nabla}$. However, the interchange of pairs symmetry of the Riemann tensor, the reality condition on $\widehat{X}_{SP}{}^{SP}$ and the Hermiticity of $\widehat{\Phi}_{ABC'D'}$ do not longer hold as these properties rely on the the cyclic identity $R_{d[abc]} = 0$. In fact, the first Bianchi identity is, in general, given by

$$\dot{R}^d{}_{[abc]} + \dot{\nabla}_{[a} \Sigma_b{}^d{}_{c]} + \Sigma_{[a}{}^e{}_b \Sigma_{c]}{}^d{}_e = 0.$$

It follows that $\widehat{X}_{A(BC)}{}^A$ does not necessarily vanish and, generically, it will depend on the torsion and its derivatives as can be seen from the last equation. Labelling, as usual, the remaning non-vanishing contractions of \widehat{X}_{ABCD}

$$\widehat{X}_{AB}{}^{AB} = 6\widehat{\Lambda}, \quad \widehat{X}_{(ABCD)} = \widehat{\Psi}_{ABCD}, \quad \widehat{X}_{A(BC)}{}^A = H_{BC},$$

where H_{BC} is a spinor which, as discussed previously, depends on the torsion and its derivatives. The explicit form of H_{BC} will not be needed. Finally, recall the general decomposition in irreducible terms of a 4-valence spinor ξ_{ABCD} :

$$\begin{aligned}\xi_{ABCD} &= \xi_{(ABCD)} + \frac{1}{2} \xi_{(AB)} P^P + \frac{1}{2} \xi P^P{}_{(CD)} \epsilon_{AB} + \frac{1}{4} \xi P^P{}^Q{}^Q \epsilon_{AB} \epsilon_{CD} \\ &\quad + \frac{1}{2} \epsilon_{A(C} \xi_{D)B} + \frac{1}{2} \epsilon_{B(C} \xi_{D)A} - \frac{1}{3} \epsilon_{A(C} \epsilon_{D)B} \xi\end{aligned}$$

where

$$\xi_{AB} \equiv \xi_{Q(AB)}{}^Q, \quad \xi \equiv \xi_{PQ}{}^{PQ}.$$

Using the above formula we get the following expressions for the irreducible decomposition of the curvature spinor \widehat{X}_{ABCD} :

$$\widehat{X}_{ABCD} = \widehat{\Psi}_{ABCD} + \widehat{\Lambda} (\epsilon_{AC} \epsilon_{BD} + \epsilon_{AD} \epsilon_{BC}) + \frac{1}{2} \epsilon_{A(C} H_{D)B} + \frac{1}{2} \epsilon_{B(C} H_{D)A}. \quad (58)$$

In order to ease the comparisons with the Levi-Civita case let

$$\widehat{Y}_{ABC'D'} = \widehat{\Phi}_{ABC'D'}. \quad (59)$$

Observe that, in contrast with the case of a Levi-Civita connection, we have that $\widetilde{\Lambda}$ is not real and $\widehat{\Phi}_{ABC'D'}$ is not Hermitian. In other words, one has that

$$\widehat{\Lambda} - \widetilde{\Lambda} \neq 0, \quad \widehat{\Phi}_{ABC'D'} - \widetilde{\Phi}_{A'B'CD} \neq 0, \quad H_{AB} \neq 0. \quad (60)$$

In fact, the right hand side of the previous equations depends on the torsion and its derivatives —see [20]. However, its explicit expression will not play any role in the discussion of this article. Having found the curvature spinors, we can derive the spinorial Ricci identities. As discussed in the previous paragraph, the modified operator $\widehat{\Xi}_{AB}$ formed from the modified commutator of covariant derivatives satisfies a version of the spinorial Ricci identities which is obtained simply by replacing the curvature spinors X_{ABCD} and $\Phi_{ABC'D'}$ by the spinors \widehat{X}_{ABCD} and $\widehat{\Phi}_{ABC'D'}$. The Ricci identities with torsion are then given by

$$\begin{aligned} \widehat{\Xi}_{AB}\xi^C &= \widehat{X}_{ABQ}{}^C \xi^Q + \Sigma_A{}^{PP'}{}_B \widehat{\nabla}_{PP'}\xi^C, & \widehat{\Xi}_{A'B'}\xi^C &= \widetilde{\Phi}_{A'B'Q}{}^C \xi^Q + \bar{\Sigma}_{A'}{}^{PP'}{}_{B'} \widehat{\nabla}_{PP'}\xi^C, \\ \widehat{\Xi}_{AB}\xi_C &= -\widehat{X}_{ABC}{}^Q \xi_Q + \Sigma_A{}^{PP'}{}_B \widehat{\nabla}_{PP'}\xi_C, & \widehat{\Xi}_{A'B'}\xi_C &= -\widetilde{\Phi}_{A'B'C}{}^Q \xi_Q + \bar{\Sigma}_{A'}{}^{PP'}{}_{B'} \widehat{\nabla}_{PP'}\xi_C, \end{aligned}$$

with \widehat{X}_{ABCD} and $\widehat{\Phi}_{ABC'D'}$ given by (58) and (59). The primed version of the last expressions can be readily identified. More importantly, the detailed version (in terms of irreducible components) of the spinorial Ricci identities become

$$\widehat{\Xi}_{AB}\xi_C = \widehat{\Psi}_{ABCQ}\xi^Q - 2\widehat{\Lambda}\xi_{(A}\epsilon_{B)C} + U_{ABCQ}\xi^Q + \Sigma_A{}^{PP'}{}_B \widehat{\nabla}_{PP'}\xi_C, \quad (61a)$$

$$\widehat{\Xi}_{(AB}\xi_{C)} = \widehat{\Psi}_{ABCQ}\xi^Q + \Sigma_{(A}{}^{PP'}{}_B \widehat{\nabla}_{PP'}\xi_{C)}, \quad (61b)$$

$$\widehat{\Xi}_{AB}\xi^B = -3\widehat{\Lambda}\xi_A + H_{AB}\xi^B + \Sigma_A{}^{PP'}{}_B \widehat{\nabla}_{PP'}\xi^B, \quad (61c)$$

$$\widehat{\Xi}_{A'B'}\xi_C = \xi^Q \widehat{\Phi}_{QCA'B'} + \bar{\Sigma}_{A'}{}^{PP'}{}_{B'} \widehat{\nabla}_{PP'}\xi_C. \quad (61d)$$

The above identities are supplemented by their complex conjugated version —keeping in mind the non-Hermiticity of $\widehat{\Phi}_{ABC'D'}$ and the non-reality of Λ as stated in (60). In the last list of identities we have defined in the quantity U_{ABCD}

$$U_{ABCD} \equiv \frac{1}{2}\epsilon_{A(C}H_{D)B} + \frac{1}{2}\epsilon_{B(C}H_{D)A}. \quad (62)$$

The Levi-Civita case can be readily recovered by setting $\Sigma_A{}^{PP'}{}_B = 0$ since the spinors H_{AB} and U_{ABCD} also vanish. Moreover, we also recover the pair interchange symmetry and the expressions presented in (60) become equalities.

B Appendix: the transition tensor and the torsion tensor

In this appendix we briefly derive the transition spinor relating a Levi-Civita connection ∇ with a connection $\widehat{\nabla}$ which is metric but not necessarily torsion-free. The general strategy behind this discussion can be found in [21]. Given two general connections $\widehat{\nabla}$ and ∇ we have that

$$(\widehat{\nabla}_a - \nabla_a)\xi^b \equiv Q_a{}^b{}_c \xi^c$$

where $Q_a{}^b{}_c$ is the transition tensor. It is well known that for the case of a Levi-Civita connection ∇ and a metric connection $\widehat{\nabla}$ one has that

$$\Sigma_a{}^c{}_b = -2Q_{[a}{}^c{}_{b]} \quad Q_{abc} = Q_{a[bc]}. \quad (63)$$

Therefore, the spinorial counterpart of the transition tensor can be decomposed as

$$Q_{AA'BB'CC'} = Q_{AA'BC}\epsilon_{B'C'} + \bar{Q}_{AA'B'C'}\epsilon_{BC} \quad (64)$$

where

$$Q_{AA'BC} \equiv \frac{1}{2}Q_{AA'(B|Q'|C)}Q'.$$

This expression allows to translate expressions containing the covariant derivative $\widehat{\nabla}$ to expressions containing ∇ and the transition spinor $Q_{AA'BC}$ as follows:

$$\widehat{\nabla}_{AA'}\xi^B = \nabla_{AA'}\xi^B + Q_{AA'}{}^B{}_Q\xi^Q, \quad \widehat{\nabla}_{AA'}\xi_C = \nabla_{AA'}\xi_C - Q_{AA'}{}^Q{}_C\xi_Q. \quad (65)$$

These expression can be extended in a similar manner to spinors of any index structure. Now, from the equations in (63) it follows that

$$Q_{acb} = -\Sigma_{a[cb]} - \frac{1}{2}\Sigma_{cab}. \quad (66)$$

Using the above equation along with the decompositions (64) and (56) we get

$$Q_{AA'BC} = -2\Sigma_{(B|AA'|C)} - 2\Sigma_{A(C|A'|B)} - 2\bar{\Sigma}_{A'(C|Q')}Q'\epsilon_{A|B}.$$

C Appendix: explicit expressions for the subsidiary equations

In Section 4.2 it was shown that the generic form of the equations in the subsidiary system is

$$\widehat{\square}\widehat{N}_{ABK} = 2\widehat{\square}_{PA}\widehat{N}^P{}_{BK} - 2\widehat{\nabla}_{AQ'}W^{Q'}{}_{BK}.$$

In this section we use the results of Appendices A and B to explicitly compute the terms $\widehat{\square}_{PA}\widehat{N}^P{}_{BK}$ and $W^{Q'}{}_{BK}$ for every zero-quantity. One has that:

$$\begin{aligned} \widehat{\square}_{PA}\widehat{\Sigma}^P{}_{B^c} &= -3\widehat{\Lambda}\widehat{\Sigma}_{AB^c} + H_{PA}\widehat{\Sigma}^P{}_{B^c} + \widehat{\Psi}_{PABG}\widehat{\Sigma}^{PG}{}_{B^c} - 2\widehat{\Lambda}\widehat{\Sigma}^P{}_{(P^c\epsilon_A)B} \\ &\quad + U_{PABQ}\widehat{\Sigma}^P{}_{(P^c\epsilon_A)B} + 2\Sigma_P{}^{QQ'}{}_A\widehat{\nabla}_{QQ'}\widehat{\Sigma}^P{}_{B^c}, \\ \widehat{\square}_{P'C'}\widehat{\Xi}_{AB}{}^{P'}{}_{D'} &= \widehat{\Xi}^Q{}_B{}^{P'}{}_{D'}\widehat{\Phi}_{QAP'C'} + \bar{\Sigma}_{P'}{}^{QQ'}{}_{C'}\widehat{\nabla}_{QQ'}\widehat{\Xi}_{AB}{}^{P'}{}_{C'} \\ &\quad + \widehat{\Xi}_A{}^{QP'}{}_{D'}\widehat{\Phi}_{AQP'C'} + \bar{\Sigma}_{P'}{}^{QQ'}{}_{C'}\widehat{\nabla}_{QQ'}\widehat{\Xi}_{AB}{}^{P'}{}_{C'} \\ &\quad - \widehat{\Lambda}\widehat{\Xi}_{ABC'D'} + \bar{H}_{P'C'}\widehat{\Xi}_{AB}{}^{P'}{}_{D'} + \bar{\Sigma}_{P'}{}^{QQ'}{}_{C'}\widehat{\nabla}_{QQ'}\widehat{\Xi}_{AB}{}^{P'}{}_{D'} \\ &\quad + \widehat{\Psi}_{P'C'D'Q'}\widehat{\Xi}_{AB}{}^{P'Q'} - \widehat{\Lambda}\widehat{\Xi}_{AB}{}^{P'}{}_{(P'\epsilon_{C'})D'} + \bar{U}_{P'C'D'Q'}\widehat{\Xi}_{AB}{}^{P'Q'} \\ &\quad + \bar{\Sigma}_{P'}{}^{QQ'}{}_{C'}\widehat{\nabla}_{QQ'}\widehat{\Xi}_{AB}{}^{P'}{}_{D'}, \\ \widehat{\square}_{PC}\widehat{\Delta}^P{}_{DBB'} &= -3\widehat{\Lambda}\widehat{\Delta}_{CDBB'} + H_{PC}\widehat{\Delta}^P{}_{DBB'} + \Sigma_P{}^{QQ'}{}_C\widehat{\nabla}_{QQ'}\widehat{\Delta}^P{}_{DBB'} + \widehat{\Psi}_{PCDQ}\widehat{\Delta}^{PQ}{}_{BB'} \\ &\quad - 2\widehat{\Lambda}\widehat{\Delta}^P{}_{(P|DBB'|\epsilon_C)D} + U_{PCDQ}\widehat{\Delta}^{PQ}{}_{BB'} + \Sigma_P{}^{QQ'}{}_C\widehat{\nabla}_{QQ'}\widehat{\Delta}^P{}_{DBB'} \\ &\quad + \widehat{\Psi}_{PCBQ}\widehat{\Delta}^P{}_{DQ}{}_{B'} - 2\widehat{\Lambda}\widehat{\Delta}^P{}_{D(P|B'|\epsilon_C)B} + U_{PCBQ}\widehat{\Delta}^P{}_{DQ}{}_{BB'} \\ &\quad + \Sigma_P{}^{QQ'}{}_C\widehat{\nabla}_{QQ'}\widehat{\Delta}^P{}_{DBB'} + \widehat{\Delta}^P{}_{DB}{}^{Q'}\widehat{\Phi}_{Q'B'PC} + \Sigma_P{}^{QQ'}{}_C\widehat{\nabla}_{QQ'}\widehat{\Delta}^P{}_{DBB'}, \\ \widehat{\square}_{P'B'}\Lambda_B{}^{P'}{}_{AC} &= \Lambda^{QP'}{}_{AC}\widehat{\Phi}_{QBP'B'} + \bar{\Sigma}_{P'}{}^{QQ'}{}_{B'}\widehat{\nabla}_{QQ'}\Lambda_B{}^{P'}{}_{AC} - 3\widehat{\Lambda}\Lambda_{BP'AC} \\ &\quad + \bar{H}_{P'B'}\Lambda_B{}^{P'}{}_{AC} + \bar{\Sigma}_{P'}{}^{QQ'}{}_{B'}\widehat{\nabla}_{QQ'}\Lambda_B{}^{P'}{}_{AC} + \Lambda_B{}^{P'Q}{}_C\widehat{\Phi}_{QAP'B'} \\ &\quad + \bar{\Sigma}_{P'}{}^{QQ'}{}_{B'}\widehat{\nabla}_{QQ'}\Lambda_B{}^{P'}{}_{AC} + \Lambda_B{}^{P'}{}_A{}^Q\widehat{\Phi}_{QCP'B'} + \bar{\Sigma}_{P'}{}^{QQ'}{}_{B'}\widehat{\nabla}_{QQ'}\Lambda_B{}^{P'}{}_{AC}. \end{aligned}$$

Moreover, one has that

$$\begin{aligned}
W[\Sigma]^{Q'}_{B^c} &\equiv Q^{Q'}_{E^E F} \widehat{\Sigma}^F_{B^c} - Q^{Q'}_{EB^F} \widehat{\Sigma}^E_{F^c}, \\
W[\Xi]^{Q}_{ABD'} &\equiv -Q^Q_{E'A^F} \widehat{\Xi}^F_{FB^{E'}D} - Q^Q_{E'B^F} \widehat{\Xi}^F_{AF^{E'}D} + \bar{Q}^Q_{E'E'F'} \widehat{\Xi}^F_{AB^{F'}D} - Q^Q_{E'D^F} \widehat{\Xi}^F_{AB^{E'}F}, \\
W[\Delta]^{Q'}_{DBB'} &\equiv Q^{Q'}_{E^F} \widehat{\Delta}^F_{DBB'} - Q^{Q'}_{ED^F} \widehat{\Delta}^F_{FBB'} - Q^{Q'}_{EB^F} \widehat{\Delta}^F_{DFB'} - \bar{Q}^{Q'}_{EB'^F} \widehat{\Delta}^F_{DBF'}, \\
W[\Lambda]^{Q}_{BAC} &\equiv -Q_{E'^G B^F} \Lambda^F_{F^{E'}AC} + \bar{Q}_{E'^G E'^F} \Lambda^F_{F'^{E'}AC} - Q_{E'^G A^F} \Lambda^F_{B^{E'}FC} - Q_{E'^G C^F} \Lambda^F_{B^E AF}, \\
W[Z]^{AA'}_{AA'} &\equiv -Q_{AA'^A} Z^{EA'} - \bar{Q}_{AA'^A} Z^{EE'},
\end{aligned}$$

where the transition spinor is understood to be expressed in terms of the reduced torsion spinor which is, in itself, a zero-quantity —see equation (28) .

D Appendix: the frame conformal Einstein field equations

The tensorial (frame) version of the standard vacuum conformal Einstein field equations are given by the following system—see e.g. [6, 5, 7, 8]:

$$\Sigma_a^c b e_c = 0, \quad (67a)$$

$$\nabla_e d^e_{abf} = 0, \quad (67b)$$

$$\nabla_c L_{db} - \nabla_d L_{bc} - \nabla_a \Xi d^a_{bcd} = 0, \quad (67c)$$

$$\nabla_a \nabla_b \Xi + \Xi L_{ab} - s g_{ab} = 0, \quad (67d)$$

$$\nabla_a s + L_{ac} \nabla^c \Xi = 0, \quad (67e)$$

$$R^c_{abd} - \rho^c_{abd} = 0, \quad (67f)$$

where $\Sigma_a^c b$ is the torsion tensor, given in terms of the connection coefficients, as

$$\Sigma_a^c b e_c = [e_a, e_b] - (\Gamma_a^c b - \Gamma_b^c a) e_c,$$

L_{ab} is the Schouten tensor, Ξ is the conformal factor, s is the concomitant of the conformal factor defined by

$$s \equiv \frac{1}{4} \nabla_a \nabla^a \Xi + \frac{1}{24} R \Xi.$$

In addition, ρ^a_{bcd} is the algebraic curvature and R^c_{dab} is the geometric curvature.

$$\begin{aligned}
\rho^a_{bcd} &= \Xi d^a_{bcd} + 2(g^a_{[c} L_{d]b} - g_{b[c} L_{d]}^a), \\
R^c_{dab} &= e_a(\Gamma_b^c d) - e_b(\Gamma_a^c d) + \Gamma_f^c d(\Gamma_b^f a - \Gamma_a^f b) + \Gamma_b^f d \Gamma_a^c f - \Gamma_a^f d \Gamma_b^c f - \Sigma_a^f b \Gamma_f^c d.
\end{aligned}$$

E Appendix: basic existence and stability theory for quasilinear wave equations

In this appendix we will give an adapted version of a theorem for quasilinear wave equations given in [15].

E.0.1 General set up

In what follows, we will consider open, connected subsets $\mathcal{U} \subset \mathcal{M}_T \equiv [0, T) \times \mathcal{S}$ for some $T > 0$ and \mathcal{S} an oriented, compact 3-dimensional manifold. Mostly, we will have $\mathcal{S} \approx \mathbb{S}^3$. On \mathcal{U} one can introduce local coordinates $x = (x^\mu) = (t, x^\alpha)$. Given a fixed $N \in \mathbb{N}$, in what follows, let

$\mathbf{u} : \mathcal{M}_T \rightarrow \mathbb{C}^N$ denote a \mathbb{C}^N -valued function. The derivatives of \mathbf{u} will be denoted, collectively, by $\partial \mathbf{u}$. We will consider quasilinear wave equations of the form

$$g^{\mu\nu}(x; \mathbf{u}) \partial_\mu \partial_\nu \mathbf{u} = \mathbf{F}(x; \mathbf{u}, \partial \mathbf{u}), \quad (68)$$

where $g^{\mu\nu}(x; \mathbf{u})$ denotes the contravariant version of a Lorentzian metric $g_{\mu\nu}(x; \mathbf{u})$ which depends smoothly on the unknown \mathbf{u} and the coordinates x and \mathbf{F} is a smooth \mathbb{C}^N -valued function of its arguments. In order to formulate a Cauchy problem for equation (68) it is necessary to supplement it with initial data corresponding to the value of \mathbf{u} and $\partial_t \mathbf{u}$ on the initial hypersurface \mathcal{S} . For simplicity, choose coordinates such that \mathcal{S} is described by the condition $t = 0$. Given two functions $\mathbf{u}_\star, \mathbf{v}_\star \in H^m(\mathcal{S}, \mathbb{C}^N)$, $m \geq 2$, one defines the ball of radius ε centred around $(\mathbf{u}_\star, \mathbf{v}_\star)$ as the set

$$B_\varepsilon(\mathbf{u}_\star, \mathbf{v}_\star) \equiv \{(\mathbf{w}_1, \mathbf{w}_2) \in H^m(\mathcal{S}, \mathbb{C}^N) \times H^m(\mathcal{S}, \mathbb{C}^N) \mid \|\mathbf{w}_1 - \mathbf{u}_\star\|_{\mathcal{S}, m} + \|\mathbf{w}_2 - \mathbf{v}_\star\|_{\mathcal{S}, m} \leq \varepsilon\}.$$

Also, given $\delta > 0$ define

$$D_\delta \equiv \{(\mathbf{w}_1, \mathbf{w}_2) \in H^m(\mathcal{S}, \mathbb{C}^N) \times H^m(\mathcal{S}, \mathbb{C}^N) \mid \delta < |\det g_{\mu\nu}(\mathbf{w}_1)|\}.$$

The basic existence and Cauchy stability theory for equations of the form (68) has been given in [15], from where we adapt the following result:

Theorem 2. *Given an orientable, compact, 3-dimensional manifold \mathcal{S} , consider the Cauchy problem*

$$\begin{aligned} g^{\mu\nu}(x; \mathbf{u}) \partial_\mu \partial_\nu \mathbf{u} &= \mathbf{F}(x; \mathbf{u}, \partial \mathbf{u}), \\ \mathbf{u}(0, x) &= \mathbf{u}_\star(x) \in H^m(\mathcal{S}, \mathbb{C}^N), \\ \partial_t \mathbf{u}(0, x) &= \mathbf{v}_\star(x) \in H^m(\mathcal{S}, \mathbb{C}^N), \quad m \geq 4, \end{aligned}$$

and assume that $g_{\mu\nu}(x; \mathbf{u}_\star)$ is a Lorentzian metric such that $(\mathbf{u}_\star, \mathbf{v}_\star) \in D_\delta$ for some $\delta > 0$. Then:

- (i) *There exists $T > 0$ and a unique solution to the Cauchy problem defined on $[0, T) \times \mathcal{S}$ such that*

$$\mathbf{u} \in C^{m-2}([0, T) \times \mathcal{S}, \mathbb{C}^N).$$

Moreover, $(\mathbf{u}(t, \cdot), \partial_t \mathbf{u}(t, \cdot)) \in D_\delta$ for $t \in [0, T)$.

- (ii) *There is a $\varepsilon > 0$ such that a common existence time T can be chosen for all initial data conditions on $B_\varepsilon(\mathbf{u}_\star, \mathbf{v}_\star) \cap D_\delta$.*

- (iii) *If the solution \mathbf{u} with initial data \mathbf{u}_\star exists on $[0, T)$ for some $T > 0$, then the solutions to all initial conditions in $B_\varepsilon(\mathbf{u}_\star, \mathbf{v}_\star) \cap D_\delta$ exist on $[0, T]$ if $\varepsilon > 0$ is sufficiently small.*

- (iv) *If ε and T are chosen as in (i) and one has a sequence $(\mathbf{u}_\star^{(n)}, \mathbf{v}_\star^{(n)}) \in B_\varepsilon(\mathbf{u}_\star, \mathbf{v}_\star) \cap D_\delta$ such that*

$$\|\mathbf{u}_\star^{(n)} - \mathbf{u}_\star\|_{\mathcal{S}, m} \rightarrow 0, \quad \|\mathbf{v}_\star^{(n)} - \mathbf{v}_\star\|_{\mathcal{S}, m} \rightarrow 0, \quad \text{as } n \rightarrow \infty,$$

then for the solutions $\mathbf{u}^{(n)}(t, \cdot)$ with $\mathbf{u}^{(n)}(0, \cdot) = \mathbf{u}_\star^{(n)}$ and $\partial_t \mathbf{u}^{(n)}(0, \cdot) = \mathbf{v}_\star^{(n)}$, it holds that

$$\|\mathbf{u}^{(n)}(t, \cdot) - \mathbf{u}(t, \cdot)\|_{\mathcal{S}, m} \rightarrow 0 \quad \text{as } n \rightarrow \infty$$

uniformly in $t \in [0, t)$ as $n \rightarrow \infty$.

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